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Hands-On Advanced Physics Laboratory

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Abstract

This comprehensive textbook provides a practical guide to advanced laboratory techniques and scientific thinking for physics students. It covers experimental design, data analysis, statistical methods, and scientific writing, with hands-on examples and model experiments.

Keywords physics, laboratory, experimental design, data analysis, scientific writing

1. Bringing Experiment to the Foreground

This book grows from a simple conviction: the principles of experimentation deserve center stage in introductory physics labs.

Physics laboratories offer the perfect setting for this approach. Their systems and theories strike that ideal balance—complex enough to be meaningful, yet simple enough that students can see the bones of experimental method beneath. The beauty of focusing on experimental principles is that it serves everyone in the room—future physicists certainly, but equally those bound for engineering, medicine, business, or the arts. Everyone benefits from understanding how we know what we claim to know.

The experimental landscape has transformed dramatically in recent decades. New instruments have played their part, but the computing revolution has fundamentally reimagined what's possible. Analyses that once demanded weeks of calculation now happen instantaneously. Computer-controlled apparatus adjusts in real-time. Data visualization reveals patterns invisible to earlier generations of scientists.

Yet beneath these technological advances, the fundamental principles remain unchanged. In fact, understanding these principles may matter more now than ever before. Modern lab setups can create a kind of experimental black box—data flows in, answers flow out, but the phenomena themselves remain hidden behind layers of processing. Without a deep understanding of what happens at each stage, invisible flaws can produce seemingly valid but meaningless results. Surrendering experimental thinking to computers is a perilous path.

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The textbook unfolds naturally through this territory. This carefully sequenced approach helps students navigate both the technological advances transforming modern labs and the timeless principles that ground experimental work. Chapter 1 establishes our experimental-centered approach to physics laboratories. Chapters 2-4 build the essential knowledge foundation—measurement theory, statistical thinking, and scientific methodology. Chapter 5 walks through practical experiment design with clarity and purpose, while Chapter 6 shows how to evaluate experimental results thoughtfully. Chapter 7 guides students through communicating their work effectively.

The appendices offer deeper dives where needed: Appendix 1 provides rigorous mathematical foundations for the Gaussian distribution and statistical analysis; Appendix 2 details the principle of least squares and its applications in data fitting; Appendix 3 introduces modern computational tools through Python and Jupyter for data analysis; and Appendix 4 presents a complete model experiment demonstrating the full experimental journey—from design through execution to final reporting.

Together, these resources bridge theoretical foundations with practical implementation, equipping students with both conceptual understanding and technical skills for modern experimental physics.

2. Approach to Laboratory Work

Important

Beyond Physics: A Universal Approach

While this textbook begins in physics labs, its scope extends far beyond. It serves as an introduction to experimental methodology that applies across all fields where we systematically study our world.

Students taking introductory physics may follow various career paths—some continuing in physics research, others pursuing different sciences, and many entering non-scientific fields entirely. Regardless of path, learning fundamental experimental principles provides valuable skills for everyone.

The text adopts a broad definition of experimentation: the complete process of identifying something in our world to study, gathering information about it, and interpreting what we find. This comprehensive view encompasses everything from molecular biologists manipulating DNA to market researchers surveying consumer toothpaste preferences.

Tip

Empowering Investigators and Informed Consumers

This book aims to serve anyone who needs to investigate aspects of the world around them or evaluate scientific claims made by others. It provides a foundation in experimental thinking that proves useful across disciplines and contexts, helping readers become both better investigators and more critical consumers of scientific information.

2.a. Understanding Scientific Knowledge and Measurement

Note

Why Everyone Needs Scientific Literacy

One might question why everyone, not just scientists, should understand how we acquire knowledge about our world. The answer lies in how experimentation permeates our lives, whether we recognize it or not.

Even non-scientists frequently need to evaluate experimental information in daily life. Professionals may need to compare competing equipment specifications, while ordinary citizens form opinions on issues like nuclear power safety, food additives, environmental concerns like global warming, or how monetary policy affects unemployment. These situations require understanding scientific experimental processes and critically assessing information reliability.

To do this effectively, we must first comprehend measurement itself. Crucially, we must recognize that measurements cannot be exact. **Uncertainty** in measurements stems from instrumental limitations or statistical variations in the measured quantity. Acknowledging this uncertainty and knowing how to estimate it allows us to properly evaluate measured values.

Warning

Avoiding Misconceptions About Scientific Claims

Beyond understanding measurement, we must address widespread misconceptions about scientific statements. These misunderstandings typically involve the authority or reliability of scientific claims. Views range from blindly accepting "scientifically proven" facts as infallible to dismissing all science as "mere theories" that can be ignored.

Neither extreme position is correct. Public discourse improves when we can appropriately evaluate scientific and technical statements on a credibility scale. Before examining how information is gathered, we must appreciate a vital but often neglected point essential for proper understanding.

Note

System vs. Model: A Crucial Distinction

This critical distinction separates the actual world being examined (the **system** under study) from the concepts and ideas (the **model**) we create after observing the system. While understanding measurements is relatively straightforward, the model concept requires elaboration.

We create ideas to represent observed system properties concisely, enabling efficient communication with shared understanding. For instance, if we were Earth's first explorers, we might repeatedly encounter similar fruit. Rather than describing each sighting separately as unrelated events, we could create the abstract concept "banana" with specific properties, facilitating more efficient communication about future meals. Beyond simple examples, models are used extensively and sophisticatedly throughout society.

In everyday communication, we often forget that many statements concern concepts rather than actual reality. Usually, this distinction doesn't matter, but sometimes it is crucial, and ignoring it leads to serious errors.

Caution

The Danger of Confusing Models with Reality

The danger arises because these two aspects of external knowledge differ fundamentally. Observations of our system belong to reality and (despite

> necessary uncertainty) can be essentially indisputable. No reasonable person would question that the Atlantic Ocean's width exceeds a living room's length. This potential incontrovertibility of observational statements can misleadingly suggest all scientific statements contain absolute truths.

> **Note**
>
> The Meaning of "Proof" in Science
>
> A common misconception is that scientists "prove" theories in an absolute sense. In reality, scientific conclusions are always provisional - they represent our best current understanding supported by evidence, but remain subject to revision. When scientists say data "supports" or "confirms" a theory, they mean the evidence aligns with the theory's predictions, not that it has achieved mathematical certainty. This distinction between provisional scientific knowledge and absolute proof is crucial for properly evaluating claims about our world.

Conceptual statements differ fundamentally from observations - they are human-created ideas designed to represent systems, not absolute truths. While carefully constructed, these models remain provisional and improvable, unlike direct observations which can provide more concrete evidence.

Misunderstanding the complementary roles of observation and concept causes much confusion in scientific debates. Climate scientists often face this when their models predict warming trends that don't immediately match year-to-year observations, forgetting that climate models represent long-term patterns rather than short-term weather events.

> **Note**
>
> Categorizing Scientific Statements
>
> All scientific statements fall into distinct categories: observations about systems, statements about models, or statements about system-model relationships. Analyzing scientific claims within these categories helps form accurate judgments.

When making scientific statements ourselves, we should use precise language. We still hear renowned scientists announce finding a "correct theory," which may be clear to those who understand such conventional language but can mislead non-scientists. Those making scientific statements should carefully monitor their language to prevent misunderstanding.

2.b. Purpose of Physics Laboratory

> **Note**
>
> Physics Labs as Training Grounds for Experimental Skills

> What connection exists between physics laboratories and broader educational goals? Though physics teaching labs serve a familiar function, we might wonder how standard laboratory experiments can introduce general experimental principles. The answer lies not in the experiments themselves but in our approach to them.

As we'll explore experimental methods further, it helps to view the subject under investigation as a system—any defined entity functioning in a specific way. We can influence systems through inputs (our control methods) and observe outputs (the system's measurable functions).

Consider various examples: A climate scientist might view Earth's climate as a system with inputs like greenhouse gas concentrations and solar radiation, while outputs include global temperature and precipitation patterns. Though we desire specific climate outcomes, we cannot directly control them—we must work through inputs, with complex relationships that climate models help us understand.

Some systems, while still complex, allow more successful control. An electrical power grid has inputs like generator operation and pricing, with outputs including power delivery and service reliability. These outputs remain determined by the system itself, not by direct management control.

How does this relate to introductory physics labs? Why not immediately address important issues like mercury contamination in fish or fossil fuel impacts on climate? The challenge is that these represent extremely complex problems with disputed evidence and interpretation. Developing skills through simpler systems provides necessary preparation.

An automobile engine represents a moderately complex system with inputs like fuel supply and ignition timing, and outputs including RPM and exhaust composition. The relationships become more predictable, though changing one input still affects multiple outputs.

A simple pendulum offers an even clearer example—a system with minimal components (string, mass, support) and straightforward inputs (string length, initial conditions) and outputs (frequency, amplitude). The connections between inputs and outputs are direct and reproducible, making fundamental experimental principles visible.

> **Tip**
>
> The Value of Simple Systems
>
> This reveals the value of introductory physics laboratories. Viewing a pendulum merely as "just a pendulum" leads to boredom. However, seeing it as a simplified version of real-world systems provides an excellent simulation environment. The

> physics laboratory offers practice with simple systems to develop expertise applicable to more complex real-world situations.

The approach matters significantly. Following rigid instructions yields limited benefits. Since real-world experimental situations vary enormously—from biological sciences dominated by random fluctuations to astronomy with precise measurement but limited control—we need general experimental principles applicable across domains.

Traditional laboratory approaches often prove inappropriate for this purpose. We should avoid viewing experiments as exercises in reproducing "correct" answers. Instead, we should objectively assess system properties and accept results as they come. Rather than following prescribed procedures, we must develop confidence in making independent experimental decisions—a crucial skill in real-world situations where guidance is rarely available.

Experiment planning deserves significant emphasis, as this stage requires substantial skill. Preliminary planning isn't a distraction from measurement but essential preparation requiring dedicated time before measurements begin.

Working within resource constraints develops important skills. Professional experimentation always faces limitations, and optimizing results within these boundaries represents a key experimental skill. Time restrictions and imperfect apparatus shouldn't be seen as defects but as realistic challenges. Good experimental evaluation requires separating valuable measurements from errors and uncertainties. Experimenters must identify error sources independently and evaluate residual uncertainty accurately—skills acquired only through realistic working conditions.

Laboratory time becomes most productive when experiments are approached as independent problem-solving opportunities. Though errors will occur, learning from direct personal experience exceeds rigidly following established procedures. Experiment outcomes matter less than learning, though skill development requires seriously pursuing optimal results.

Clear communication of laboratory findings is as crucial as the research itself. Scientific work gains value only when effectively shared, making reporting a core professional skill beyond mere writing. Thorough documentation and constructive critique create vital learning opportunities that reveal their full benefit in retrospect.

2.c. Glossary

2.d. Problems

Exercise 1:

For each scenario, identify whether the statement refers to a system or a model: a) The temperature readings from a weather station b) The equations describing planetary motion c) A computer simulation of protein folding d) The actual protein molecules in a test tube

Exercise 2:

Design a simple experiment to test how string length affects a pendulum's period. Include: a) The system under study b) Input variables you'll control c) Output variables you'll measure d) Potential sources of uncertainty

Exercise 3:

Find a scientific claim in a news article and analyze it by: a) Identifying the system being studied b) Describing how measurements were likely made c) Assessing potential sources of uncertainty d) Evaluating whether the conclusions follow from the evidence

Exercise 4:

A student measures a block's length five times: 10.2 cm, 10.4 cm, 10.3 cm, 10.5 cm, 10.3 cm a) Calculate the mean length b) Estimate the uncertainty range c) Explain possible sources of variation d) Suggest how to reduce uncertainty

Exercise 5:

Evaluate this statement: "The model predicted temperatures within 2°C of actual measurements." a) What does this tell us about the model's accuracy? b) What additional information would help assess the model's quality? c) How might the model be improved? d) Why can't we say the model is "proven"?

Exercise 6:

Choose a common household appliance and analyze it as an experimental system: a) Identify its key components b) Describe inputs you can control c) Describe outputs you can measure d) Explain how you might model its behavior

3. Measurement and Uncertainty

3.a. Understanding the Measuring Process

> When you can measure what you are speaking about and express it in numbers, you know something about it; but when you cannot express it in numbers, your knowledge remains meager and unsatisfactory. – Lord Kelvin

Measurement lies at the heart of our scientific understanding. Though perhaps overstated, this sentiment captures an essential truth - proper measurement forms the foundation of meaningful experimentation.

Note

At its core, measurement involves comparing an object or phenomenon with some reference standard. These reference standards (meters, kilograms, seconds, etc.) must be universally agreed upon, which is why international organizations establish and maintain measurement standards.

Let's begin with a simple example to understand the fundamental nature of measurement. Imagine measuring the height of a coffee mug with a ruler marked in millimeters. You might report "87 mm," but does this mean the mug is exactly 87.00000… mm tall? Of course not. What you're really doing is determining that the height falls within some interval - perhaps between 86.5 mm and 87.5 mm.

Through this dual process of approaching from above and below, we identify an interval - the smallest range within which we're confident the true value lies. This reveals measurement's essential nature: we don't determine exact values but rather intervals of possibility.

Important

When reporting measurements, we must specify both the central value and the interval width. This determination requires careful judgment based on numerous factors: scale precision, lighting conditions, object definition, visual acuity, and more.

We must assess each situation individually rather than following oversimplified rules (like assuming uncertainty equals half the smallest scale division). A well-defined object under perfect conditions might allow precision well beyond the smallest marked division, while a poorly defined object might create uncertainty spanning several divisions.

3.b. Understanding Digital Readouts and Rounding

Digital instruments present their own interpretive challenges. When a digital multimeter displays "3.82 V," what exactly does this mean? The answer depends on the instrument's design.

Tip

Most commonly, the reading indicates the value falls between 3.815 V and 3.825 V - the instrument rounds to the nearest displayed digit.

However, some digital timers might operate differently, showing "10:15" for any time between exactly 10:15:00 and 10:15:59. Each instrument type requires understanding its specific operation.

This highlights a broader concept: rounding introduces its own form of uncertainty. When we write $\pi = 3.14$, we understand this isn't exactly true. Rather, we mean the value lies between 3.135 and 3.145.

Warning

"Rounding uncertainty" may seem trivial, but it can significantly impact calculations, especially when:

1. Many rounded values accumulate errors throughout a calculation
2. Two nearly equal values are subtracted, making relative errors much larger
3. High powers are involved, amplifying small errors

With modern calculators, it's wise to maintain extra digits throughout calculations, rounding appropriately only at the final step. Similarly, statements like "measured to the nearest millimeter" inadequately convey measurement uncertainty, as they establish only minimum bounds for the measurement interval.

3.c. Absolute and Relative Uncertainty

Measurements should represent the range within which we believe the true value lies. For instance, we might determine a tabletop's length lies between 152.7 cm and 153.1 cm. While this interval representation is entirely valid, we often restate it as 152.9 ± 0.2 cm.

This uncertainty value (± 0.2 cm) represents the **absolute uncertainty** of our measurement. However, the significance of any uncertainty depends on the measurement's magnitude. An uncertainty of ± 0.2 cm would be:

Context	Impact of ± 0.2 cm
Measuring microchip components	Substantial
Measuring furniture	Acceptable
Measuring astronomical distances	Negligible

Table 1: Impact of Uncertainty in Different Contexts

To better evaluate a measurement's quality, we use **relative uncertainty**, defined as:

$$\text{Relative Uncertainty} = \frac{\text{Absolute Uncertainty}}{\text{Measured Value}} \tag{1}$$

For our tabletop example:

$$\text{Relative Uncertainty} = \frac{0.2 \ \text{ cm}}{152.9 \ \text{ cm}} = 0.0013 \ \text{ or } 0.13\% \tag{2}$$

Note

This relative value provides a more meaningful assessment of precision. We often call this the **precision** of our measurement. Absolute uncertainty carries the same units as the measurement itself, while relative uncertainty is a dimensionless ratio.

3.d. Identifying Systematic Errors

The uncertainties discussed so far arise from natural limitations in measurement processes. However, another category - systematic errors - affects all measurements in a consistent way.

These systematic errors, particularly **calibration errors**, require vigilance. Always check instrument zeros before measurement and verify calibration when possible.

Warning

Don't be misled by sophisticated digital displays with multiple "precise" digits. When measuring current, all ammeters introduce their own internal resistance into the circuit. This resistance creates a voltage drop that alters the actual current flowing through the circuit, meaning the displayed current differs from what would flow without the meter.

For example, a digital multimeter might display a "stable" current reading of 1.23 A, but the actual circuit current could be significantly different due to the meter's internal resistance. High-quality meters minimize this effect with very low internal resistance (often called "burden voltage"), but it can never be completely eliminated.

Similarly, when measuring voltage, the meter draws some current to operate, potentially affecting the circuit's behavior. Always consider how your measuring instrument might be altering the very quantity you're trying to measure.

Approach all instruments with healthy skepticism, recognizing that:

1. Displayed precision often exceeds actual accuracy
2. The act of measurement can change the system being measured
3. Understanding instrument specifications (like internal resistance) is crucial for proper interpretation

3.e. Calculating Uncertainty in Derived Quantities

Rarely does a single measurement complete our work. Usually, we need to calculate some quantity based on multiple measurements or apply mathematical operations to our measured values.

When calculating uncertainties in derived quantities, we will focus on finding the maximum possible uncertainty by considering the absolute values of all component uncertainties. This approach ensures we account for the worst-case scenario where all uncertainties combine to produce the largest possible error in our final result.

3.f. Uncertainty in Single-Variable Functions

Consider a measured quantity x with uncertainty $\pm\delta x$, and a calculated result $z = f(x)$. The maximum possible uncertainty in z is determined by considering how much z could change when x varies by $\pm\delta x$.

For example, if $z = \frac{x}{x^2+4}$:

$$\delta z = \left| \frac{4 - x^2}{\left(x^2+4\right)^2} \right| \delta x \tag{3}$$

Let's examine several common function types:

3.f.i. *Powers and Roots:*

This reveals an important principle: the relative uncertainty in the result equals the relative uncertainty in the measurement multiplied by the power. This applies to both positive powers (multiplication) and negative powers (division/roots).

3.f.ii. *Exponential Functions:*

3.f.iii. *Logarithmic Functions:*

3.f.iv. *Trigonometric Functions:*

3.g. Uncertainty in Multi-Variable Functions

When dealing with functions of multiple variables, we calculate the maximum possible uncertainty by taking the sum of the absolute values of all contributing uncertainties. This approach ensures we account for the worst possible case where all uncertainties combine to maximize the final uncertainty.

3.g.i. *Sum and Difference of Variables:*

Figure 1: Interactive demonstration of uncertainty propagation in addition and subtraction.

3.g.ii. *Products and Quotients:*

3.h. General Approach for Multi-Variable Functions

3.h.i. *Complex Functions:*

For more complex functions, break them down into simpler components and apply the chain rule, always using absolute values to ensure maximum uncertainty:

3.i. Understanding Significant Figures: Purpose Over Rules

When working with measurements, significant figures serve a critical purpose that goes beyond mere rule-following. They communicate the quality and reliability of your measurements to others. While textbooks often present lengthy lists of rules about significant figures, it's more valuable to understand their fundamental purpose.

At their core, significant figures represent **the digits that are known with certainty, plus one additional digit that represents your best estimate**. This approach emerges naturally from the measurement process itself.

Consider how you might record a measurement from a graduated cylinder. When the liquid level falls between markings, you don't simply write down the nearest mark. Instead, you estimate to one digit beyond what the scale directly shows. That estimated digit—the last significant figure—carries valuable information about your measurement.

Rather than memorizing a complex set of rules about zeroes and calculations, focus first on the fundamental principle: significant figures reflect the precision of measurement. When you understand this purpose, many of the rules become intuitive rather than arbitrary.

```
# Let's write a simple function to estimate significant figures in a
measurement
def count_sig_figs(measurement_str):
    """Estimate the number of significant figures in a measurement"""
    # Remove any units that might be present
    measurement_str = measurement_str.split()[0]

    # Handle scientific notation
    if 'e' in measurement_str.lower():
        base, exponent = measurement_str.lower().split('e')
        return count_sig_figs(base)

    # Count significant digits according to basic rules
    digits = ''.join(c for c in measurement_str if c.isdigit())
    if '.' in measurement_str:
        # With decimal point, trailing zeros are significant
        # Remove leading zeros
        digits = digits.lstrip('0')
        return len(digits)
    else:
        # Without decimal point, trailing zeros might not be significant
        # This is ambiguous without more context
        digits = digits.lstrip('0')
        # Remove trailing zeros as they're ambiguous
        digits = digits.rstrip('0')
        return len(digits)

# Test with some examples
examples = ["12.34", "0.0056", "1200", "1200.0", "0.1200"]
for example in examples:
    print(f"Measurement: {example}, Significant figures:
{count_sig_figs(example)}")
```

When propagating significant figures through calculations, focus on mastering the multiplication rule first:

- The result of multiplication or division should have the same number of significant figures as the measurement with the fewest significant figures.

Other rules for addition, logarithms, and special functions become easier to learn once you've established this foundation.

3.i.i. *A Practical Approach to Zeroes:*

Zeroes often cause the most confusion when determining significant figures. Instead of memorizing rules, consider where the number came from:

- A measurement of 4.70 mL from a graduated cylinder has three significant figures because you estimated the last digit (confirming zero tenths)
- A measurement of 4.7 mL has two significant figures, indicating you didn't estimate beyond the tenths place
- A measurement of 470 mL is ambiguous without context—it could have two or three significant figures

When teaching or learning significant figures, focusing on their purpose—communicating measurement quality—provides a more meaningful framework than simply memorizing rules. This understanding helps you make appropriate judgments when recording and working with experimental data.

Calculations often produce more digits than are justified by our measurement precision. We must quote results sensibly.

3.j. Glossary

3.k. Problems

Exercise 7:

I measure a window pane's width between 68.3 cm and 68.9 cm. Express this as a central value with uncertainty, and calculate the relative uncertainty.

Exercise 8:

A digital scale displays 235.8 g when weighing a sample. If the scale rounds to the nearest 0.1 g, what is the absolute uncertainty?

Exercise 9:

If my measuring tape has absolute uncertainty ± 0.5 mm, what's the shortest distance I can measure while maintaining relative uncertainty below 0.5%?

Exercise 10:

I measure the dimensions of a rectangular sheet as (25.4 ± 0.2) cm $\times$ (18.6 ± 0.2) cm. What is the absolute uncertainty in the calculated area?

Exercise 11:

A capacitance value is calculated using $C = \frac{\varepsilon_0 A}{d}$ with measurements:

- Area $A = (0.025 \pm 0.001)\ \text{m}^2$
- Distance $d = (0.5 \pm 0.02)$ mm
- $\varepsilon_0 = 8.85 \times 10^{-12}$ F \/ m (exact)

Calculate the value and uncertainty of C.

Exercise 12:

When determining wave velocity using $v = \lambda f$, I measure wavelength $\lambda = (0.75 \pm 0.05)$ m and frequency $f = (440 \pm 5)$ Hz. Find the absolute and relative uncertainty in velocity.

Exercise 13:

A value is reported as 583.2417 ± 0.15. Rewrite this with appropriate significant figures.

Exercise 14:

The resistance of a wire is measured at two temperatures:

- $R_1 = (125.3 \pm 0.4)\,\Omega$ at $T_1 = 20°$ C
- $R_2 = (138.1 \pm 0.4)\,\Omega$ at $T_2 = 50°$ C

Calculate the temperature coefficient of resistance and its uncertainty using $\alpha = \frac{R_2 - R_1}{R_1 (T_2 - T_1)}$.

4. Statistics of Measurement

4.a. Understanding Random Variation

When we make measurements, we often observe that repeated measurements of the same quantity show random variations. This is a fundamental aspect of experimental science that we must understand and account for.

Note

These variations can arise from many sources:

- Environmental fluctuations (temperature, pressure, humidity changes)
- Instrument limitations (finite resolution, electronic noise)
- Human factors (reaction time variations, reading parallax)
- Quantum effects (in some cases, such as radioactive decay)

Consider measuring the radioactivity of a sample. Even with perfect equipment, the number of counts in a fixed time interval will vary randomly due to the inherent stochastic nature of radioactive decay. Similarly, optical measurements might show fluctuations due to air currents causing refractive index variations or thermal effects causing mechanical instabilities in the apparatus.

4.b. The Gaussian Distribution

When we make many measurements of the same quantity, the results often follow a bell-shaped curve known as the Gaussian or normal distribution. This distribution is fundamental to understanding measurement uncertainty.

The mathematical form of the Gaussian distribution is:

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{4}$$

where μ is the population mean and σ is the population standard deviation.

These percentages are crucial for understanding measurement uncertainty. The 68-95-99.7 rule (sometimes called the empirical rule) provides a quick way to assess the likelihood that a measurement falls within certain bounds of the true value.

This distribution allows us to make meaningful statements about our measurements. For example, if we measure a length multiple times and find a mean of 10.5 cm with a standard deviation of 0.1 cm, we can say:

Important

We are approximately 68% confident that any single measurement will fall between 10.4 cm and 10.6 cm, and 95% confident it will fall between 10.3 cm and 10.7 cm.

4.c. Sample Statistics and Population Parameters

When we make measurements, we're typically working with a sample from a larger population of possible measurements. Understanding the relationship between sample statistics and population parameters is essential.

The **sample mean** is calculated as:

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \tag{5}$$

The **sample standard deviation** is:

$$s = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N - 1}} \tag{6}$$

where N is the sample size and x_i are individual measurements.

Note the $(N - 1)$ in the denominator, known as Bessel's correction, which provides an unbiased estimate of the population standard deviation.

The **standard error of the mean** tells us how precisely we've determined the population mean. As our sample size increases, this uncertainty decreases as $1/\sqrt{N}$.

> **Warning**
>
> Don't confuse **standard deviation** (variation in individual measurements) with **standard error** (uncertainty in the mean). The standard deviation describes the spread of the data, while the standard error describes how precisely we know the mean.

4.c.i. *Distinction Between Standard Deviation and Standard Error:*

This distinction is crucial and frequently misunderstood:

- **Standard Deviation (s)**: Describes the variability of individual measurements around the sample mean. It tells us about the inherent scatter in our data.
- **Standard Error of the Mean ($s_m = s/\sqrt{N}$)**: Describes the uncertainty in our estimate of the population mean. It tells us how precisely we know the "true" value.

For example, if we measure the same quantity 25 times and get $s = 2.0$ units:

- The standard deviation remains 2.0 units (describing individual measurement scatter)
- The standard error of the mean is $2.0/\sqrt{25} = 0.4$ units (our uncertainty in the mean)

4.d. *Propagation of Statistical Uncertainty*

When we calculate derived quantities from multiple measurements, we need to understand how the uncertainties combine. The propagation formulas depend on whether we're dealing with estimated uncertainties or statistical uncertainties.

4.d.i. *General Error Propagation Rules:*

For a function $z = f(x, y)$ where x and y are independent variables with standard deviations σ_x and σ_y:

$$\sigma_z^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 \tag{7}$$

Note

These rules assume the measurements are independent and the uncertainties are random (not systematic). For systematic errors, the propagation rules are different.

4.d.ii. *Statistical vs. Estimated Uncertainties:*

When combining different types of uncertainties (e.g., some estimated, some statistical), we need to ensure compatibility. If one uncertainty represents a 68% confidence interval (1 standard deviation) and another represents outer limits (∼100% confidence), they cannot be directly combined using the standard propagation formulas.

4.e. Central Limit Theorem and Sampling

The Central Limit Theorem explains why the Gaussian distribution is so prevalent in measurement science. It states that the distribution of sample means approaches a normal distribution as the sample size increases, regardless of the shape of the original population distribution.

This theorem justifies our use of Gaussian statistics even when individual measurements might not follow a perfect Gaussian distribution.

4.f. Identifying and Handling Outliers

Sometimes our measurements include values that seem unusually different from the others. These outliers require careful consideration and systematic analysis.

4.f.i. *Chauvenet's Criterion:*

Chauvenet's criterion provides a statistical method for identifying potential outliers. The criterion states that a measurement should be rejected if the probability of obtaining a deviation as large or larger is less than $1/(2N)$, where N is the total number of measurements.

Procedure for Chauvenet's Criterion:

1. Calculate the sample mean ($\bar{x}$) and standard deviation (s)
2. For each measurement x_i, calculate the deviation: $d_i = | x_i - \bar{x} |$
3. Express this as a number of standard deviations: $t_i = d_i / s$
4. Find the probability that a measurement would deviate by t_i or more standard deviations (using Gaussian tables)

5. If this probability is less than $1/(2N)$, the measurement is a candidate for rejection

Example: For $N = 10$ measurements, reject if probability < 0.05 (about 2σ) For $N = 20$ measurements, reject if probability < 0.025 (about 2.2σ)

Important Guidelines for Outlier Rejection:

1. **Never reject data simply because it doesn't fit expectations**
2. **Check for obvious experimental errors first** (misreading, equipment malfunction)
3. **Apply statistical criteria systematically, not arbitrarily**
4. **Document all rejected data with reasoning**
5. **Remember that outliers sometimes indicate important physics**

4.f.ii. *Systematic Approach to Outliers:*

When handling potential outliers, follow this systematic approach:

1. **Verify the measurement** was recorded correctly
2. **Check for obvious experimental problems** (equipment malfunction, environmental disturbance)
3. **Apply statistical assessment** (such as Chauvenet's criterion)
4. **Document thoroughly** the reasoning for any rejected measurements
5. **Never reject data simply because it doesn't fit expectations**

The probability guidelines from the Gaussian distribution help us make these decisions:

- Outside 2σ limits: 5% probability (might be legitimate outlier)
- Outside 3σ limits: 0.3% probability (likely candidate for rejection)
- Outside 4σ limits: 0.006% probability (very likely experimental error)

4.g. *Confidence Intervals and Uncertainty Statements*

Understanding how to make proper uncertainty statements is crucial for communicating experimental results.

4.g.i. *Confidence Intervals:*

A confidence interval provides a range of values that likely contains the true population parameter. For a 95% confidence interval of the mean:

$$\mathrm{CI}_{95\%} = \bar{x} \pm 1.96 \times \frac{s}{\sqrt{N}} \tag{8}$$

This means we're 95% confident the true population mean lies within this range.

4.g.ii. *Proper Uncertainty Statements:*

When reporting results:

- **68% confidence**: $\bar{x} \pm s_m$ (1 standard error)
- **95% confidence**: $\bar{x} \pm 1.96 \times s_m$ (≈ 2 standard errors)
- **99.7% confidence**: $\bar{x} \pm 3 \times s_m$ (3 standard errors)

4.h. Sample Size Effects

The size of our sample dramatically affects the reliability of our statistical estimates.

4.h.i. *Effect on Standard Error:*

The standard error of the mean decreases as $1/\sqrt{N}$:

- To halve the uncertainty in the mean, need 4 times as many measurements
- To reduce uncertainty by factor of 10, need 100 times as many measurements

4.h.ii. *Reliability of Standard Deviation Estimates:*

For small samples, our estimate of the population standard deviation is quite uncertain. The standard deviation of the standard deviation is approximately:

$$\sigma_s \approx \frac{\sigma}{\sqrt{2(n-1)}} \quad (9)$$

This means:

- With $N = 5$: our s estimate has ∼35% uncertainty
- With $N = 10$: our s estimate has ∼25% uncertainty
- With $N = 25$: our s estimate has ∼15% uncertainty

Minimum Sample Size Guidelines:

- For meaningful statistics: $N \geq 10$
- For reliable standard deviation estimates: $N \geq 20$
- For precise confidence intervals: $N \geq 30$

4.i. Combining Different Types of Uncertainty

In practice, we often need to combine uncertainties that have different statistical meanings (e.g., some estimated, some statistical).

4.i.i. *Making Uncertainties Compatible:*

If combining a statistically-based uncertainty (68% confidence) with an estimated range uncertainty (∼100% confidence), we need to make them compatible:

- Convert estimated range to ∼68% confidence by multiplying by 0.67
- Or convert statistical uncertainty to ∼100% range by multiplying by 1.5

4.i.ii. *Root Sum of Squares:*

For independent uncertainties of the same confidence level:

$$\sigma_{total} = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \dots} \quad (10)$$

This assumes the uncertainties are:

- Independent (not correlated)
- Random (not systematic)
- Of the same confidence level

4.j. Distribution Shapes and Assumptions

While we often assume Gaussian distributions, real measurements may deviate from this ideal.

4.j.i. *When Gaussian Assumptions Fail:*

- **Small number statistics**: For counting experiments with few counts, use Poisson statistics
- **Skewed distributions**: May occur with certain measurement processes
- **Multiple peaks**: Could indicate multiple measurement modes or systematic effects

4.j.ii. *Checking Gaussian Assumptions:*

Simple tests for Gaussian behavior:

- Plot histogram of residuals from the mean
- Check if ∼68% fall within 1 standard deviation
- Look for systematic patterns in the data

4.k. Practical Measurement Strategy

A well-planned measurement strategy can minimize uncertainties and improve data quality.

Before starting measurements:

1. **Estimate expected uncertainty** based on instrument resolution and known fluctuations
2. **Determine required sample size** based on target precision
3. **Choose measurement sequence** to minimize systematic effects
4. **Plan for outlier detection** and handling procedures

4.l. Glossary

4.m. Problems

Exercise 15:

Construct a histogram of the following measurements (in cm): 12.3, 12.5, 12.4, 12.6, 12.3, 12.4, 12.5, 12.4, 12.3, 12.5

Exercise 16:

Calculate the mode, median, mean, standard deviation, and standard error for the measurements in Problem 1.

Exercise 17:

For the measurements in Problem 1, find the 95% confidence interval for the true value.

Exercise 18:

Using Chauvenet's criterion, determine if any of the measurements in Problem 1 should be considered outliers.

Exercise 19:

If we measure a rectangle's length as (12.5 ± 0.5) cm and width as (18.6 ± 0.2) cm, what is the uncertainty in the perimeter?

Exercise 20:

How many measurements would be needed to reduce the standard error by a factor of 2 compared to Problem 1?

5. The Nature of Scientific Thinking

5.a. The Interplay Between Observation and Understanding

> **Important**
>
> Scientific activity follows certain patterns that emerge naturally from the challenges we face when trying to understand the world around us. To grasp how scientific thinking works, let's imagine we're developing a completely new field of study from first principles.

When confronted with an unfamiliar phenomenon, our initial impulse is to ask: "What causes this?" This fundamental question has driven investigations into everything from light diffraction to radioactivity, from superconductivity to pulsars. Today, we continue asking this same question about elementary particles, climate change, cancer, and countless other phenomena.

> **Note**
>
> While asking about causes seems natural, we should recognize that these questions are fundamentally about relationships between observable variables. Rather than pursuing abstract notions of causation that might lead us into philosophical quagmires, scientists look for consistent patterns of relationship.

For instance, when investigating electrical conductivity, we might observe that current flow depends strongly on the potential difference across a conductor but shows no relationship to whether the conductor points north-south or east-west. This observation, though seemingly elementary to modern eyes, represents exactly the kind of relationship-finding that drives scientific progress.

During initial investigations of new phenomena, we focus on identifying which variables matter and which don't. By determining these significant relationships, we narrow our focus to manageable dimensions and create the foundation for both experimental work and theoretical understanding.

> **Tip**
>
> Interestingly, this initial stage of science allows us to make relatively definitive statements, since we're describing direct observations. This partly explains why scientific activities gain a reputation for revealing "scientific truth." However, this certainty applies primarily to the basic identification of relationships. As we move beyond this foundational stage, we enter realms that involve much greater uncertainty and interpretation.

5.b. Models: The Conceptual Heart of Science

After identifying significant variables, we progress to a more sophisticated level of understanding by developing models. To appreciate what models are and how they function, consider a simple example:

The critical insight here is that we're dealing with two entirely different categories:

1. The actual physical wall that needs painting
2. A conceptual rectangle constructed from mathematical definitions

Warning

We often overlook this distinction because the concept of rectangles is so familiar that we instinctively assess whether a wall is "rectangular enough" for our calculation to be useful. But imagine if we couldn't make this assessment – if, like a blind person, we could only measure the base and one side without verifying angles or other properties. In such cases, we might calculate an area with little relevance to the actual wall (if, for instance, the wall was shaped like a parallelogram).

To avoid such errors, we would need to systematically check whether our conceptual rectangle matches the actual wall by comparing multiple properties: straightness of sides, right-angle corners, equality of diagonals, and so on. Only after confirming sufficient correspondence between our mental model and physical reality could we confidently use the calculated area for practical purposes.

Important

This distinction between reality and mental constructs lies at the heart of scientific thinking. In all scientific endeavors, we find ourselves navigating between two realms:

- The physical world and our observations of it
- Conceptual constructs built from definitions and assumptions

These constructs are called **models**, and they pervade both scientific and everyday thinking. The painter envisioning a rectangular wall, the botanist categorizing a flower within a species, and the economist analyzing a national economy using equations – all are using models to represent reality.

Note

Models serve as shorthand descriptions of systems, providing frameworks for thought, communication, calculation, and further investigation. However, we must remember their fundamental nature: models are invented concepts, not reality itself. While we construct them to correspond as closely as possible to the physical world, no model can ever be an exact replica of reality. A wall isn't actually a rectangle; a wheel isn't actually a circle. At best, a model's properties may be similar to reality's properties, and a model's usefulness depends on how well these properties align.

5.c. Testing Models Against Reality

For a model to be scientifically useful, it must be testable against observation. This requirement distinguishes scientific models from other forms of thought. A proposition about "how many angels can dance on a pinhead" falls outside science not because it's necessarily meaningless, but because it can't be tested against experience. Such ideas may still have value as mathematical, philosophical, aesthetic, or ethical propositions – they simply aren't scientific.

5.d. Refining Models Through Iteration

When these discrepancies become significant at our required level of precision, we must modify our model. We might adjust angles or dimensions, hoping that these refinements will improve the match between model and reality. Even with such adjustments, the model remains a conceptual construct, and the calculated area belongs to the model, not to the physical wall itself.

> **Important**
>
> This principle applies throughout science. We should feel free to modify our models whenever necessary, since they're simply tools we've created to help understand reality. Our only consideration should be improving the model's utility. Because it's likely impossible to create a description that perfectly captures every aspect of physical reality, the continuous refinement and eventual replacement of models is a natural part of scientific progress.

This ongoing refinement process defines much of what scientists do, whether in "pure" research, technological development, or social sciences. While challenging work, this process builds on generations of previous efforts. In our professional lives, we're fortunate if we can make even small improvements to existing models. Major revisions or entirely new models are rare achievements, often worthy of Nobel Prizes.

> **Tip**
>
> Yet we needn't be fixated on perpetual model improvement. Though no model perfectly captures reality, many models correspond sufficiently well for practical purposes. In such cases, we can proceed confidently with our work, remembering to periodically verify the model's continued adequacy. Rather than thinking in terms of "right" or "wrong" models, we should consider whether a model is "adequate," "suitable," or "appropriate" for our specific purposes.

5.e. The Historical Development of Scientific Models

Consider Louis de Broglie's 1924 proposal of matter's wave properties, published before direct observation of electron diffraction, or Enrico Fermi's conception of the neutrino, proposed nearly four decades before experimental detection. There is no singular "scientific method" – rather, ideas and observations advance together, sometimes with one leading, sometimes the other.

> **Important**

> What remains constant, regardless of developmental sequence, is the fundamental scientific activity: comparing models with reality through experiment.

We haven't discussed how entirely new theoretical frameworks emerge. Sometimes existing ideas undergo gradual refinement, achieving better correspondence with observation without fundamentally changing (like the Ptolemaic system of planetary epicycles). Other times, progress requires radical reconceptualization (as with Einstein's general relativity or Schrödinger's wave mechanics).

When determining a well's depth by dropping a stone, Einstein's general relativity isn't necessary – Newtonian mechanics suffices. We reserve more sophisticated models for circumstances that demand them, like predicting Mercury's orbital peculiarities. Einstein's theory doesn't invalidate Newton's for everyday applications; it simply provides better correspondence with reality at higher precision or in extreme conditions.

> **Tip**
>
> Generally, we choose theories based on adequacy for our purposes. When higher precision becomes necessary, we introduce appropriate refinements (unless working at knowledge frontiers where improved theories don't yet exist).

5.f. Making Precise Comparisons Between Models and Reality

Let's now examine how we practically compare models with physical systems. Vague conceptual comparisons won't suffice; we need explicit, quantitative methods. This typically requires quantitative observation of the system alongside mathematical specification of the model.

However, to make this vague notion useful for detailed comparison with reality, we need mathematical precision. We might measure the elastic band's extension as a function of load, collecting data like that shown in Table 4.1.

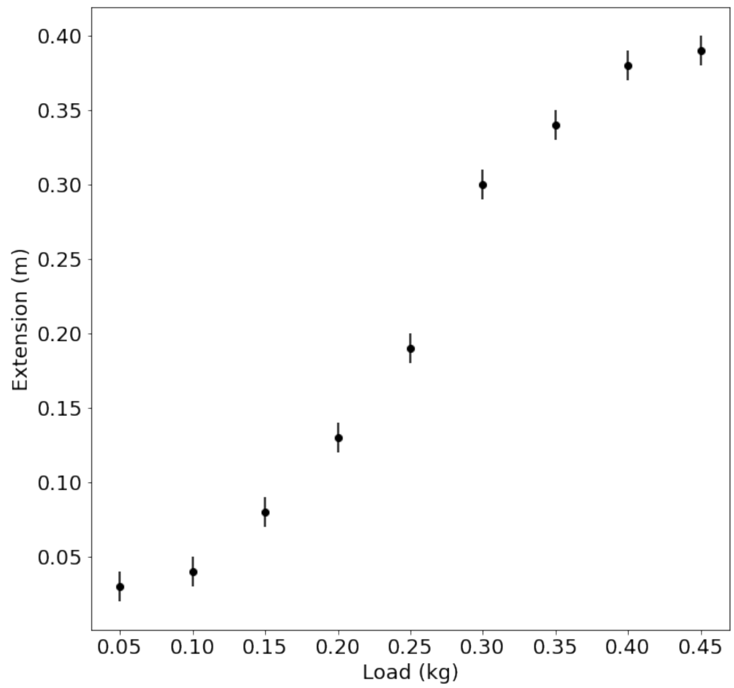

Figure 2: Extension of an elastic band as a function of load, showing central values (data points) and uncertainty ranges (error bars). This graphical representation reveals patterns that are difficult to discern from tabular data alone.

Note

While we've now collected measurements, a table of numbers doesn't readily reveal patterns or relationships. Visual representation helps tremendously. By plotting these measurements on a graph (Figure 4.1), including both central values and uncertainty ranges, we can more easily judge the system's behavior.

At this stage, we've completed only the observation phase. Our next task is constructing a model to represent the system.

5.g. *Approaches to Model Construction*

Let's examine each approach.

5.g.i. *Empirical Models:*

Smooth Curve Fitting: A more sophisticated approach involves drawing a smooth curve through observed points (Figure 2). This assumes the system's behavior is continuous and regular despite measurement uncertainty and scatter.

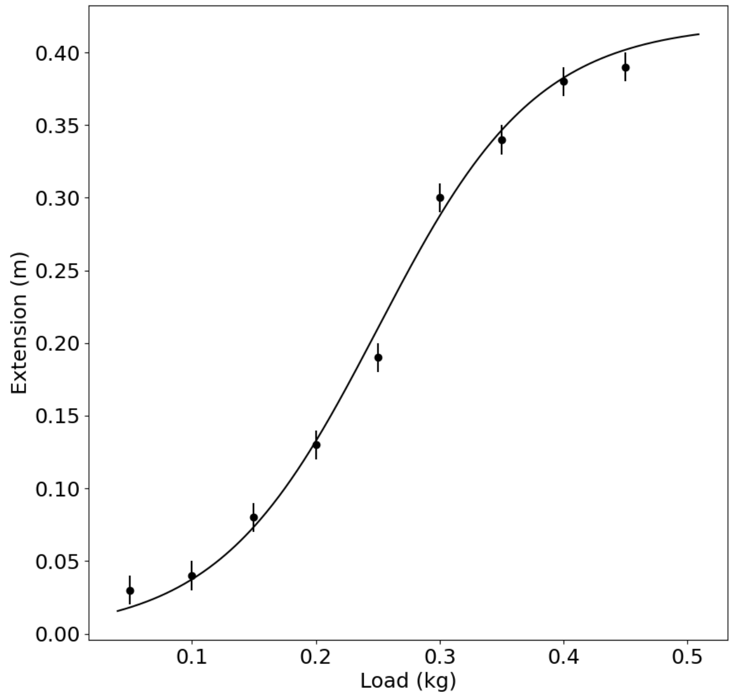

Figure 3: A smooth curve fitted through experimental observations, representing an empirical model of the system's continuous behavior.

> **Warning**
>
> This assumption often holds for physical systems (like planetary motion), but responsibility for making this judgment rests with the experimenter based on knowledge of the system.

The smooth curve approach offers practical benefits, particularly for interpolation and extrapolation. If we need to estimate extension at a load between measured values, the curve provides a systematic method (Figure 3).

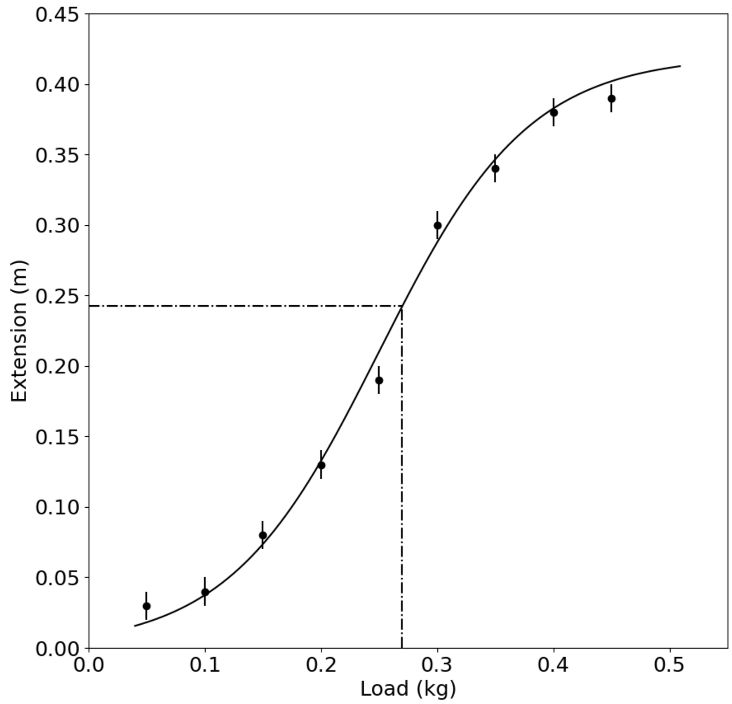

Figure 4: Interpolation: using a smooth curve to estimate values between measured data points.

Caution

Similarly, we might attempt to estimate values beyond our measurement range (Figure 4), though extrapolation is inherently less reliable than interpolation. We should have strong reasons to believe the system's behavior remains consistent beyond measured ranges, as smooth behavior within measured regions doesn't guarantee similar behavior beyond them (Figure 5).

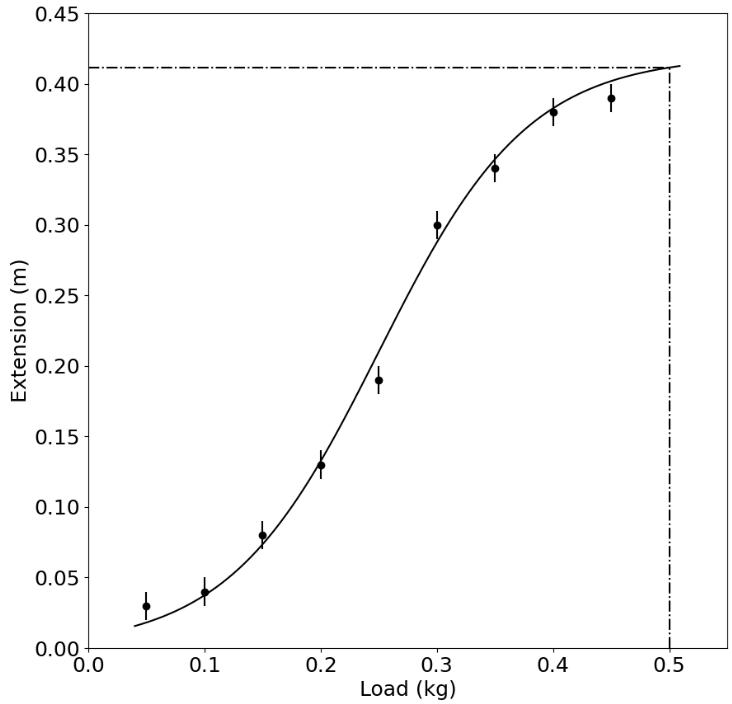

Figure 5: Extrapolation: extending the smooth curve beyond the range of measured data points.

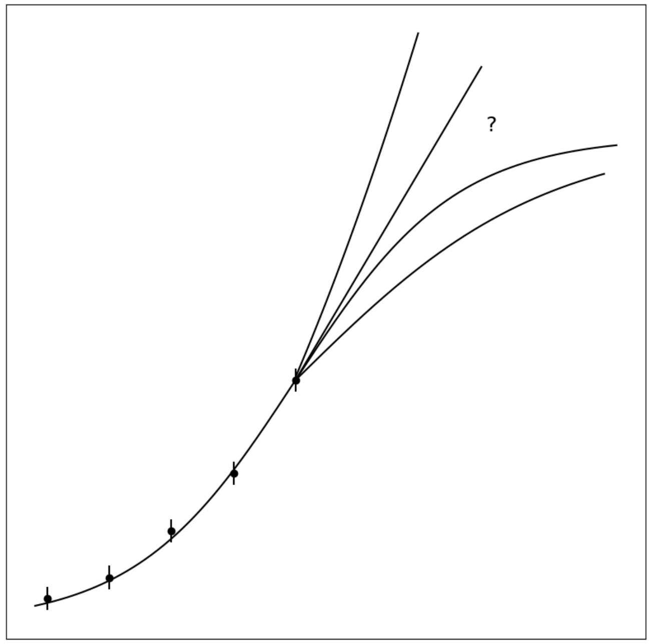

Figure 6: The danger of extrapolation: system behavior may change dramatically beyond the measured range, causing extrapolated predictions to diverge from reality.

Mathematical interpolation/extrapolation methods can perform these estimates without physically drawing curves, but they still depend fundamentally on assumptions about the system's regularity.

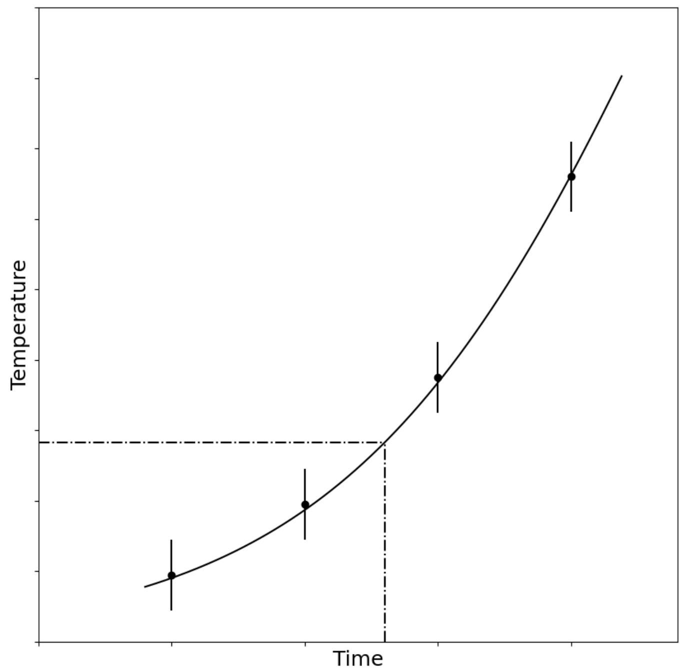

Figure 7: A cautionary example: these temperature measurements represent noontime readings on successive days. Interpolating between points would incorrectly estimate midnight temperatures.

A common but problematic practice is connecting measured points with straight-line segments (Figure 7). Computer graphics often do this automatically. But such representations satisfy neither the requirements of observation (they're not data points) nor modeling (they don't represent our conceptual understanding of the system).

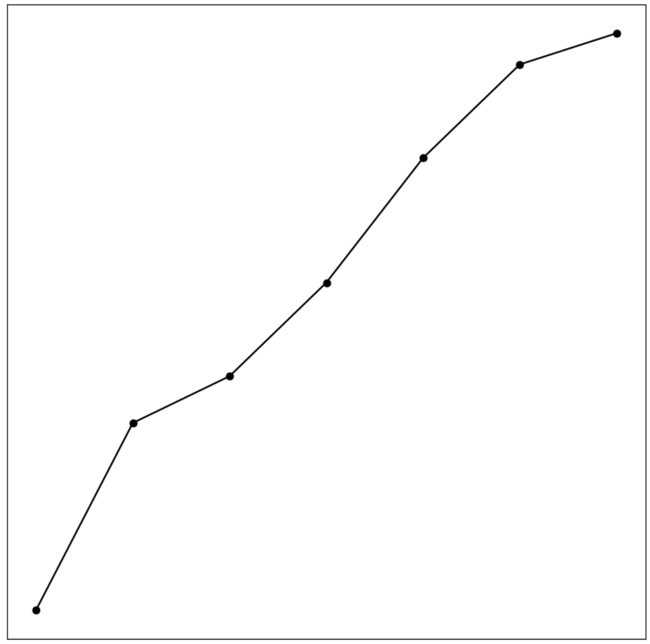

Figure 8: A problematic representation: connecting data points with straight-line segments neither represents raw observations nor a meaningful model of system behavior.

Empirically derived functions can serve as useful mathematical models, enabling interpolation and extrapolation with varying degrees of precision. However, we must remember that these functions' validity as models depends on how well they capture the system's actual behavior.

Note

Extrapolation particularly highlights model limitations. We can accurately predict sunset times weeks ahead because astronomical models are excellent, but weather forecasting becomes increasingly uncertain with time, and stock market prediction remains nearly impossible. The model's quality determines prediction reliability.

5.g.ii. *Theoretical Models:*

Let's illustrate this with an example:

For a theoretical approach, we'd begin with fundamental principles. We might hypothesize constant gravitational acceleration:

$$a = 9.8 \text{ m } \backslash / \text{ s}^2 \tag{11}$$

This hypothesis immediately incorporates assumptions (like neglecting air resistance) that begin our model construction process. These assumptions might or might not make it a “good” model – that determination awaits experimental verification.

```
Through mathematical derivation (integration), we obtain:

$$v = 9.8t \text{ (assuming } v=0 \text{ at } t=0\text{)}$$

And (taking downward as the positive direction):

$$x = \frac{9.8}{2}t^2 \text{ (assuming } x=0 \text{ at } t=0\text{)}$$

Rearranging to express time as a function of distance:

$$t = \left(\frac{1}{4.9}\right)^{1/2}x^{1/2}$$

:::{note}
Throughout this derivation, each assumption becomes part of our model.
The final equation represents a property of our model, not necessarily
of reality. We must next determine how well this theoretical prediction
matches actual measurements.
```

5.h. Comparing Theoretical Models with Experimental Results

A more effective approach uses visual comparison. Figure 9 shows: (a) a graph of our experimental measurements as points, (b) our theoretical model as a continuous curve, and (c) both superimposed for direct comparison.

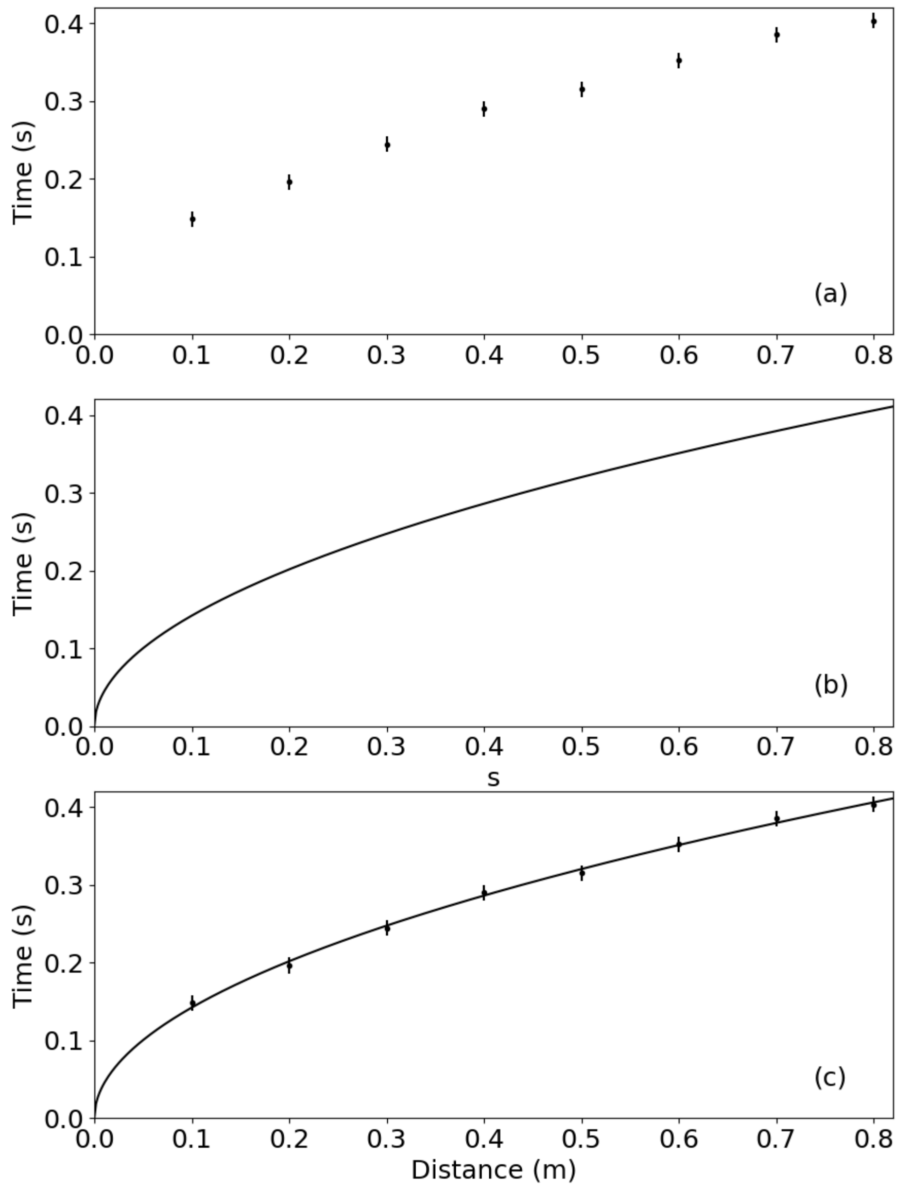

Figure 9: Comparing model predictions with experimental observations: (a) experimental measurements as data points, (b) theoretical model as a continuous curve, and (c) superposition of both for direct comparison.

Important

> This visual comparison allows us to judge overall correspondence between model and reality. We can immediately see whether they agree, where discrepancies exist, and their magnitude relative to measurement uncertainty.

This approach clarifies what we can legitimately claim after an experiment. We can state that model and system behavior correspond (or don't) to a certain extent – not that a theory is "true," "correct," or "wrong." Such terminology misrepresents the nature of models. Better to describe models as "satisfactory," "good enough," or "appropriate" for particular purposes.

Yet Einstein's theory doesn't invalidate Newton's for everyday applications – it simply provides a more comprehensive model with greater correspondence at extreme scales or precisions. Most people don't measure well depths using relativity theory! We choose models based on adequacy for our specific purpose, introducing refinements only when necessary.

> **Note**
>
> This perspective offers an interesting philosophical insight: even when model and system appear to correspond perfectly, we can only claim that, at our current precision level, we haven't detected discrepancies. We can be more definitive when finding disagreement – we can confidently state a model is inadequate if discrepancies significantly exceed measurement uncertainty.

Modern computing has transformed model comparison. While drawing graphs for complex functions once presented major difficulties, computers now display experimental measurements alongside theoretical predictions instantly. Nevertheless, understanding fundamental comparison principles remains essential, both for situations without computers and for ensuring meaningful interpretation of computer-generated results.

5.i. Linear Analysis: A Powerful Technique

Consider our free-fall time equation:

$$t = \left(\frac{1}{4.9}\right)^{1/2} x^{1/2} \tag{12}$$

Plotting this directly against measurements would create a parabolic curve, making visual assessment difficult. However, if we plot t versus $x^{1/2}$ instead, our theoretical relationship becomes linear:

$$t = 0.4515 \times x^{1/2} \tag{13}$$

Which follows the form:

$$\text{vertical variable} = \text{slope} \times \text{horizontal variable} \tag{14}$$

Where:

- vertical variable = t

- horizontal variable = $x^{1/2}$
- slope = 0.4515

```
Alternative transformations could work equally well — plotting $t^2$
versus $x$ instead of $t$ versus $x^{1/2}$ would also yield a straight
line with different slope. The choice depends on convenience and which
approach provides clearer comparison for a particular experiment.

## Determining Unknown Constants

:::{admonition} The Challenge of Unknown Parameters
:class: note
Often our theoretical models contain unknown constants that must be
determined experimentally. For example, when testing Hooke's law (that
spring extension is proportional to load), we might not know the spring
constant:

$$x = \text{constant} \times W$$
```

After measuring extension versus load and plotting the results (Figure 11a), how do we represent this model? The equation actually represents an infinite family of straight lines passing through the origin, with slopes representing all possible spring constant values.

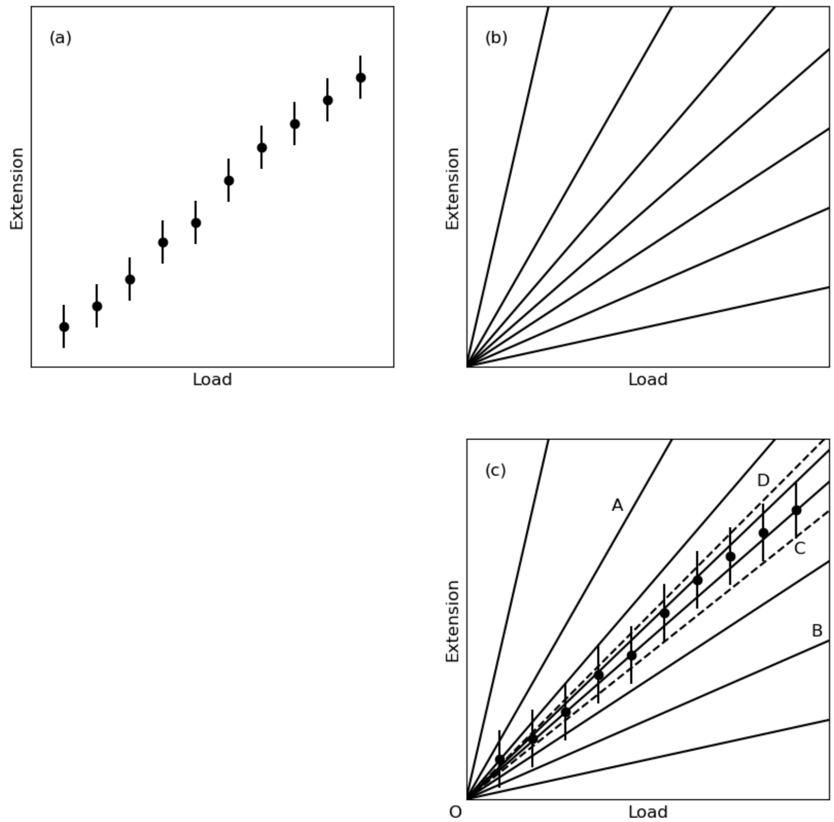

Figure 10: Determining unknown constants from graphical analysis: (a) experimental data plotted, (b) family of possible model lines, (c) identifying the range of lines consistent with experimental uncertainties.

This graphical approach offers significant advantages beyond simply testing model validity. Consider measuring electrical resistance from voltage-current measurements. We could calculate $R = V/I$ for each measurement pair and average the results, but this algebraic approach can introduce serious errors:

Warning

1. It doesn't allow us to visually assess model validity (whether V-I relationship is actually linear)
2. It cannot handle data scatter effectively
3. It cannot detect or compensate for systematic issues like unexpected intercepts or non-linearity at certain ranges

With a graphical approach, even with scattered data, we can confidently determine resistance from the slope of a line that best represents the overall trend. If measurements show an unexpected intercept or deviate from linearity in certain regions, we can still extract reliable resistance values from the linear portion, unaffected by these discrepancies.

Important

Purely algebraic calculations would incorporate values from all measurements regardless of their relationship to the underlying model, potentially introducing significant errors. The graphical method makes discrepancies immediately visible and allows informed judgment about their significance.

Importantly, this approach lets us obtain accurate parameter values even without knowing the source of discrepancies between model and system. We need only identify discrepancies and ensure they don't contaminate our results; investigating their causes can come later.

Note

While we've focused on determining constants from slopes, straight-line graphs actually provide two independent pieces of information – slope and intercept. This allows experiments to determine two separate quantities within a model, a capability we'll explore further in later chapters.

5.j. Problems

5.k. Glossary

Exercise 21:

Consider a pendulum experiment where the period T is measured at different lengths L. The measurements yield the following data:

- L = 0.25 m, T = 1.01 s
- L = 0.50 m, T = 1.42 s
- L = 0.75 m, T = 1.73 s
- L = 1.00 m, T = 2.01 s

The theoretical model predicts $T = 2\pi\sqrt{\frac{L}{g}}$, where g is the acceleration due to gravity.

Plot the data in a way that creates a straight-line graph and determine the value of g from the slope.

Exercise 22:

A scientist proposes that the current I in a circuit varies with voltage V according to the relationship $I = aV + bV^2$ where a and b are constants. The following measurements are collected:

- V = 1.0 V, I = 3.2 mA
- V = 2.0 V, I = 7.3 mA
- V = 3.0 V, I = 12.4 mA
- V = 4.0 V, I = 18.5 mA

Describe two different methods to test whether this model adequately describes the data, and determine the constants a and b.

Exercise 23:

A material's resistivity ρ is measured at various temperatures T, yielding the data:

- T = 20°C, ρ = 1.72 × 10^{-8} Ω·m
- T = 40°C, ρ = 1.85 × 10^{-8} Ω·m
- T = 60°C, ρ = 1.97 × 10^{-8} Ω·m
- T = 80°C, ρ = 2.10 × 10^{-8} Ω·m

Assuming a linear relationship between ρ and T, estimate the resistivity at 100°C. Discuss the limitations of this extrapolation and what additional information would increase its reliability.

Exercise 24:

A chemical reaction produces gas at a rate that varies with time. The volume V of gas collected at different times t is measured:

- t = 10 s, V = 12.5 mL
- t = 20 s, V = 23.8 mL
- t = 30 s, V = 33.9 mL
- t = 40 s, V = 43.1 mL
- t = 50 s, V = 51.6 mL
- t = 60 s, V = 59.4 mL

Examine the data to determine whether a linear model ($V = kt$) or a model with decreasing rate ($V = A\left(1 - e^{-kt}\right)$) better describes the reaction. Justify your conclusion using appropriate graphical analysis.

Exercise 25:

In an experiment to measure Young's modulus E for a wire, a student plots elongation ΔL versus applied force F and obtains a straight line that doesn't pass through the origin. The theoretical relationship is $\Delta L = \frac{FL}{AE}$ where L is the wire's length and A is its cross-sectional area.

Identify three possible sources of discrepancy between the theoretical model and the experimental data, and explain how each might affect the results.

Exercise 26:

The following relationship is proposed for a gas at constant temperature: $PV^n = C$ where P is pressure, V is volume, and n and C are constants.

Explain how to transform this equation to create a linear relationship and describe how you would analyze experimental P-V data to determine the values of n and C.

Exercise 27:

Two models are proposed for the relationship between a projectile's range R and its launch angle θ:

- Model A: $R = k\sin(2\theta)$
- Model B: $R = k\sin(\theta)\cos(\theta)$

Without performing calculations, discuss whether it's possible to experimentally distinguish between these models. What approach would you take?

Exercise 28:

A student measures the refractive index n of a solution at different concentrations c:

- c = 0.1 mol/L, n = 1.34
- c = 0.3 mol/L, n = 1.36
- c = 0.5 mol/L, n = 1.38
- c = 0.7 mol/L, n = 1.41
- c = 0.9 mol/L, n = 1.44

The student needs to determine the refractive index at c = 0.4 mol/L. Discuss the assumptions involved in interpolating this value and evaluate whether linear interpolation is appropriate.

Exercise 29:

A simple model suggests that for small angles, a pendulum's period T is independent of its amplitude. A student measures the period at different amplitudes and obtains:

- Amplitude = 5°, T = 1.51 s
- Amplitude = 10°, T = 1.52 s
- Amplitude = 15°, T = 1.53 s
- Amplitude = 20°, T = 1.55 s
- Amplitude = 25°, T = 1.57 s

Discuss how this data could guide model refinement. Propose a refined model that might better describe the relationship between period and amplitude.

Exercise 30:

A physicist discovers that a quantity y depends on a variable x but doesn't know the form of the function. She collects the following data:

- x = 1, y = 1.0
- x = 2, y = 1.4
- x = 3, y = 1.7
- x = 4, y = 2.0
- x = 5, y = 2.2

Describe how she could determine whether the relationship is: (a) linear, (b) logarithmic, (c) square root, or (d) exponential. What transformations would she apply to test each possibility?

6. Designing Experiments: Principles and Methods

6.a. Introduction to Experimental Design

In the previous chapter, we explored the various ways researchers compare models with real-world systems. The diversity we encountered suggests a crucial insight: there is no universal approach to planning experiments. The techniques and methodologies researchers employ necessarily depend on specific circumstances and objectives.

Despite this diversity, certain fundamental principles remain valid across virtually all experimental situations. Perhaps most important among these is keeping our experimental purpose clearly in mind: **the fundamental requirement in experimentation, regardless of what else is happening, is to compare the properties of a physical system with the properties of one or more theoretical models**.

6.b. Testing an Existing Model

Remember that determining whether a model is appropriate for a given system must be based on experimental evidence. We aren't attempting to decide whether models are "true" or "false" in some absolute sense—all models are imperfect approximations. Rather, we need to determine if a particular model is adequate for our specific purposes at our desired level of precision.

> **Warning**
>
> If conditions change or greater precision becomes necessary, we must reconsider the model's adequacy. As discussed previously, graphical approaches typically provide the most effective way to test physical models. Ideally, we want to plot the model's behavior alongside our experimental observations of the system's behavior, which requires some preparation.

Since conventional graphs are two-dimensional, we initially need to limit ourselves to examining relationships between two variables at a time. When dealing with multiple input variables, we can simplify by holding all but one constant while studying how the output variable depends on the remaining input variable. After completing this analysis, we can adjust one of the previously fixed variables and repeat the process. Through successive measurements of this kind, we can construct a comprehensive picture of the system's behavior.

> **Note**
>
> This approach assumes we can hold input variables constant independently of one another. When this isn't possible, more sophisticated techniques become necessary, which we'll touch on later.

For now, assuming we're working with a single input variable (either because only one exists or because we've isolated one by controlling the others), the procedure is straightforward: measure how the output variable changes with the input variable, then plot these measurements for comparison with the model's predictions. As noted earlier, the advantages of linear representation are so significant that we'll focus primarily on transforming data into straight-line form.

6.c. *Converting Equations to Straight-Line Form*

6.c.i. *Basic Transformations:*

Consider a function describing the time of fall for an object:

$$t = 0.4515x^{1/2} \quad \text{(in meters and seconds)} \tag{15}$$

To represent this in linear form:

$$\text{vertical variable} = \text{slope} \times \text{horizontal variable} + \text{intercept} \tag{16}$$

We might choose:

$$\text{vertical variable} = t \tag{17}$$

$$\text{horizontal variable} = x^{1/2} \tag{18}$$

$$\text{slope} = 0.4515 \tag{19}$$

$$\text{intercept} = 0 \tag{20}$$

> **Note**
>
> There's no single formula for these transformations. The most effective approach is keeping the target form clearly in mind while rearranging the original equation until we achieve the desired structure.

Multiple valid transformations often exist for a given equation. The function above could be equivalently expressed as:

$$x^{1/2} = \frac{1}{0.4515}t \tag{21}$$

$$t^2 = 0.2309x \tag{22}$$

$$x = 4.905t^2 \tag{23}$$

While convention often suggests plotting input variables horizontally and output variables vertically, there's no strict requirement to do so. Choose the representation that best serves our analytical purposes.

6.c.ii. *Practical Considerations:*

For example, when analyzing the period of a physical pendulum, the equation is given by:

$$T = 2\pi\sqrt{\frac{I}{mgd}} \tag{24}$$

Where:

- T = period of oscillation
- I = moment of inertia about the pivot point
- m = mass of the pendulum
- g = acceleration due to gravity
- d = distance from pivot to center of mass

For a compound pendulum with multiple masses at different positions, the moment of inertia becomes more complex:

$$I = \sum_{i=1}^{n} m_i \left(r_i^2 + \frac{k_i^2}{12} \right) \tag{25}$$

Where:

- m_i = mass of each component
- r_i = distance from pivot to center of each component
- k_i = length of each component (for extended objects)

Warning

We might be tempted to plot T versus $\sqrt{\frac{1}{d}}$ for different configurations, but this would require calculating the complex moment of inertia for each data point and introduce compounded uncertainties.

A better approach: square both sides of the original equation to get:

$$T^2 = 4\pi^2 \frac{I}{mgd} \tag{26}$$

Then plot T^2 versus $\frac{1}{d}$ for a fixed configuration. This gives a straight line with slope $4\pi^2 \frac{I}{mg}$. After measuring the slope, we can calculate the moment of inertia using:

$$I = \frac{mg \times \text{slope}}{4\pi^2} \tag{27}$$

Important

This method simplifies data collection and analysis while providing direct insight into the system's physical properties. By postponing the calculation of the moment of inertia until after finding the slope, we reduce error propagation and gain a clearer understanding of the pendulum's behavior.

This principle—plot variables in their simplest form and leave arithmetic for the final calculation—serves well in experimental design.

6.c.iii. *Working with Compound Variables:*

Converting this to linear form using single-variable functions of h and T proves impossible. However, using compound variables makes it possible. Starting by squaring both sides:

$$T^2 = \frac{4\pi^2(h^2 + k^2)}{gh} \tag{28}$$

Multiplying both sides by h:

$$T^2 h = \frac{4\pi^2(h^2 + k^2)}{g} \tag{29}$$

Rearranging to isolate h^2:

$$h^2 = \frac{g}{4\pi^2} T^2 h - k^2 \tag{30}$$

This gives us a linear equation where:

- Vertical variable = h^2
- Horizontal variable = $T^2 h$
- Slope = $g/4\pi^2$
- Intercept = $-k^2$

Compound variables also prove valuable with multiple input variables. When measuring specific heat using flow calorimetry, the heat balance equation is:

$$Q = mC\Delta T \tag{31}$$

Where Q is heat generation rate, m is mass flow rate, C is specific heat, and ΔT is temperature difference.

Tip

Rather than plotting ΔT versus $1/m$ (with separate curves for different Q values) or ΔT versus Q (with separate curves for different m values), we could plot $m\Delta T$ versus Q. This creates a single graph incorporating both input variables simultaneously, with slope C, enabling efficient model testing and parameter determination.

If plotting with compound variables reveals unexpected patterns (scattered data or nonlinearity), we can always revert to plotting individual variable pairs to investigate further.

6.c.iv. *Logarithmic Transformations:*

Logarithmic plotting applies to simple power relationships too. For:

$$y = x^n \tag{32}$$

Taking logarithms:

$$\log y = n \log x \tag{33}$$

Plotting log y versus log x (a "log-log plot") yields a straight line with slope n.

For instance, if measurements follow $y = x^{1.8}$ rather than $y = x^2$, plotting y versus x^2 would show systematic deviation from linearity without revealing the true relationship. A log-log plot would still produce a straight line, immediately indicating a power relationship, with the slope revealing the actual exponent (1.8).

We'll explore log-log plotting further when discussing empirical model construction in the next chapter.

6.d. Step-by-Step Experimental Planning

The planning process includes:

1. **Identify system and model**: This seemingly obvious step can be surprisingly challenging. The phenomenon under study is often surrounded by measurement apparatus, obscuring the fundamental system. If we struggle to identify the system, ask: "What entity's properties does the model describe?"

Similarly, clearly define the model's limitations. When studying falling objects, will we account for air resistance? Neglecting air resistance isn't irresponsible—it's defining one aspect of the model. The experiment itself will reveal whether this simplification is justified at the desired precision level.

1. **Select variables**: Typically, one quantity presents itself as the natural output variable. If there's only one input variable, selection is straightforward. With multiple input variables, identify the primary independent variable and vary others in discrete steps.
2. **Transform the equation**: Put the model equation into straight-line form as described earlier. Remember, multiple valid transformations usually exist. Choose one that serves our purposes effectively. When the equation contains unknown parameters to be determined experimentally, structure the transformation to place these unknowns in the slope rather than the intercept whenever possible. Intercepts are more susceptible to systematic errors from instrument defects.
3. **Determine variable ranges**: Plan for an input variable range spanning at least a factor of 10. Wider ranges provide better basis for comparing system and model behaviors. While we can't directly control output variable ranges, carefully consider instrument limitations. Circuit components have maximum current ratings, materials have elastic limits, and sensors have operating ranges. Perform trial measurements to determine input variable ranges that avoid damaging equipment or exceeding measurement capabilities.
4. **Consider experimental precision**: Begin with a target precision level for the final result. This guides measurement method selection. A request to "measure g using a pendulum" is meaningless without specifying whether we need 10% precision (achievable with simple equipment in minutes) or 0.01% precision (requiring sophisticated apparatus and days of work).

With a clear precision goal—say, measuring g within 2%—we can work backward to determine requirements for each component measurement. For a pendulum experiment, if we need g within 2%, we might aim for uncertainties in length (ℓ) and period-squared (T^2) below 1% each.

If we can measure length with ± 1mm uncertainty, the minimum acceptable length measurement would be:

$$\frac{0.001 \text{ m}}{\ell} = 0.01 \tag{34}$$

$$\ell = 0.1 \quad \text{m} \tag{35}$$

Similarly, if timing uncertainty is ± 0.2s, and period measurement requires 0.5% precision (for 1% in T^2), the minimum timing interval would be:

$$\frac{0.2 \text{ s}}{t} = 0.005 \tag{36}$$

$$t = 40 \quad \text{seconds} \tag{37}$$

This analysis helps ensure all measurements contribute meaningfully to the desired final precision. If any measurement appears limited to uncertainties exceeding the target, we'll need either more precise measurement methods or a revised precision goal.

The complete experiment design process is illustrated in Appendix A4 with a sample experiment.

Warning

While this planning may seem excessive for simple laboratory exercises, it represents the minimum preparation required for serious research. Resist the temptation to rush into measurements and figure out analysis later—developing good planning habits now will serve us well throughout our scientific careers.

6.e. Designing Experiments Without Existing Models

Even without detailed theoretical understanding, empirical models prove extremely valuable. They help organize thinking about complex systems and enable mathematical operations like interpolation, extrapolation, and forecasting.

Tip

In model-free situations, experiment design becomes more straightforward if input variables can be isolated. Simply measure the output variable across suitable ranges of input variables to construct a comprehensive picture of system behavior. When input variables can't be isolated, more complex challenges arise, as discussed in a later section.

Even without established theories, consider any available hints about potentially applicable functions, testing them against our observations. One powerful technique for obtaining such hints is dimensional analysis.

6.f. Dimensional Analysis

This approach can't determine dimensionless constants (like π), but it reveals functional relationships between variables.

For example, analyzing the velocity (v) of waves on a string under tension (T) with mass per unit length (m):

$$v \propto T^a m^b \tag{38}$$

Dimensionally:

- v: LT^{-1}
- T (tension): MLT^{-2}
- m: ML^{-1}

Therefore:

$$LT^{-1} = \left(MLT^{-2}\right)^a \left(ML^{-1}\right)^b = M^{a+b} L^{a-b} T^{-2a} \tag{39}$$

Matching powers:

- For M: 0 = a+b
- For L: 1 = a-b
- For T: −1 = −2a

Solving gives $a = \frac{1}{2}$, $b = -\frac{1}{2}$, yielding:

$$v = (\text{dimensionless constant}) \times \sqrt{\frac{T}{m}} \tag{40}$$

6.g. *Difference-Type Measurements*

6.g.i. *Null-Effect Measurements in Physical Sciences:*

For steel wires, we might cut a single wire in half, load one piece while leaving the other unloaded, and measure the difference in length. Both experience identical temperature variations, but only one responds to loading. This approach reveals small effects that would otherwise be lost in environmental noise.

> **Warning**
>
> Always check system performance both with and without the influence we're studying. Wilson's humorous observation is worth remembering: "It has been conclusively proved by numerous tests that the beating of drums and gongs during a solar eclipse causes the sun's brightness to return."

6.g.ii. *Control Groups in Biological Sciences:*

This approach often requires refinements like placebo treatments and double-blinding (keeping both experimenters and subjects unaware of group assignments) to prevent psychological effects from contaminating results.

> **Note**
>
> Such experimental designs—pairing treatment groups with carefully matched control groups—are foundational in biological research, whether studying

> carcinogenic food additives in mice or music education effects on academic performance.

6.h. Observational Studies with Uncontrollable Variables

In such cases, careful observational techniques become crucial. With well-defined systems and models (like celestial mechanics), precise measurements may still allow meaningful conclusions—determining that general relativity better explains Mercury's orbit than Newtonian mechanics, for instance.

The best approach: meticulous sampling procedures. Create artificial null-effect measurements by constructing treatment groups under the influence we're studying and control groups exempt from it but otherwise matched as closely as possible.

> **Important**
>
> The validity of such studies depends entirely on sampling quality. Effects are often subtle enough that different sampling approaches can yield contradictory conclusions. This reality explains why public policy debates featuring scientific components often present competing "scientific" evidence—different sampling approaches can support opposing conclusions.

When facing such complexity, conventional concepts like "proof" require modification. Mathematical theorems can be proven from axioms, and some physical measurements are certain enough to be considered "proven" (the moon is closer than the sun). But in complex systems with probabilistic effects, "proof" gives way to **correlation**—statistical relationships between variables that differ fundamentally from direct cause-effect relationships but remain valid for identifying influencing factors.

We'll examine correlation analysis further in Chapter 6 when discussing experimental evaluation.

6.i. Glossary

6.j. Problems

For problems 6-23, state which variables or combinations of variables should be plotted to verify the proposed relationship, and explain how to determine the unknown parameter(s) from the graph (slope, intercept, etc.).

Exercise 31:

A research paper claims the terminal velocity of a skydiver depends solely on the skydiver's mass and gravitational acceleration. Evaluate the reasonableness of designing an experiment to test this hypothesis.

Exercise 32:

A projectile is launched with initial velocity v at angle α to the horizontal. Its range may depend on the projectile's mass, initial velocity, launch angle, and gravitational acceleration. Determine the functional form of this relationship through dimensional analysis.

Exercise 33:

The internal pressure in a soap bubble depends on the liquid's surface tension and the bubble's radius. Using dimensional analysis, determine the relationship between these variables.

Exercise 34:

A torsional oscillator's period depends on the support's torsional stiffness coefficient (torque per unit angular displacement) and the moment of inertia of the oscillating object. Find the functional relationship between these quantities.

Exercise 35:

The central deflection of a beam with circular cross-section, supported at both ends and loaded at its center, depends on the applied force, distance between supports, beam radius, and the material's Young's modulus. Use dimensional analysis to determine the relationship.

Exercise 36:

The position of an object under constant acceleration follows:

$$s = \frac{1}{2}at^2 \quad (41)$$

where s and t are measurable. Determine the acceleration a.

Exercise 37:

The fundamental vibration frequency of a stretched string is given by:

$$f = \frac{1}{2\ell}\sqrt{\frac{T}{m}} \quad (42)$$

where f, ℓ, and T can be measured. Determine m.

Exercise 38:

The exit velocity of an ideal fluid flowing through an opening in a tank follows:

$$v = \sqrt{\frac{2P}{\rho}} \tag{43}$$

where v and P are measurable. Determine fluid density ρ.

Exercise 39:

A conical pendulum's period is described by:

$$T = 2\pi\sqrt{\frac{\ell \cos \alpha}{g}} \tag{44}$$

where T and α are measurable, and ℓ is fixed and known. Determine g.

Exercise 40:

The deflection of a cantilever beam follows:

$$d = \frac{4W\ell^3}{Yab^3} \tag{45}$$

where d, W, and ℓ are measurable, while a and b are fixed, known values. Determine Young's modulus Y.

Exercise 41:

The height of capillary rise in a tube follows:

$$h = \frac{2\sigma}{\rho g R} \tag{46}$$

where h and R are measurable, and ρ and g are known constants. Determine surface tension σ.

Exercise 42:

The ideal gas law states:

$$pV = RT \tag{47}$$

where p and T are measurable, and V is fixed and known. Determine gas constant R.

Exercise 43:

The Doppler frequency shift for a moving source follows:

$$f = f_0 \frac{v}{v - v_0} \tag{48}$$

where f and v_0 are measurable quantities, and f_0 is a known constant. Determine velocity v.

Exercise 44:

Thermal expansion of a solid follows:

$$\ell = \ell_0 (1 + \alpha \Delta T) \tag{49}$$

where ℓ and ΔT are measurable, and ℓ_0 is unknown but constant. Determine coefficient of expansion α.

Exercise 45:

Snell's law of refraction states:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{50}$$

where θ_1 and θ_2 are measurable angles, and n_1 is a known constant. Determine refractive index n_2.

Exercise 46:

The thin lens equation states:

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'} \tag{51}$$

where s and s' are measurable. Determine focal length f. Compare two possible plotting methods and explain which is preferable.

Exercise 47:

The resonant frequency of a parallel LC circuit follows:

$$\omega = \frac{1}{\sqrt{LC}} \tag{52}$$

where ω and C are measurable. Determine inductance L.

Exercise 48:

Coulomb's law for electrostatic force is:

$$F = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \quad (53)$$

where F and r are measurable, while q_1 and q_2 are fixed, known values. Describe how to verify the form of this relationship.

Exercise 49:

The force between parallel current-carrying conductors follows:

$$F = \frac{\mu_0}{4\pi} \frac{i_1 i_2 \ell^2}{r^2} \quad (54)$$

where F, i_1, i_2, and r are measurable quantities, while μ_0 and ℓ are constants. Describe how to verify this relationship.

Exercise 50:

The charge remaining on a discharging capacitor follows:

$$Q = Q_0 e^{-t/RC} \quad (55)$$

where Q and t are measurable, and R is known and fixed. Determine capacitance C.

Exercise 51:

The impedance of a series RC circuit follows:

$$Z = \sqrt{R^2 + \frac{1}{\omega^2 C^2}} \quad (56)$$

where Z and ω are measurable. Determine resistance R and capacitance C.

Exercise 52:

The relativistic mass variation with velocity follows:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (57)$$

where m and v are measurable. Determine rest mass m_0 and speed of light c.

Exercise 53:

The wavelengths in the Balmer series of hydrogen follow:

$$\frac{1}{\lambda} = R\left(\frac{1}{4} - \frac{1}{n^2}\right) \tag{58}$$

where λ and n are measurable. Determine Rydberg constant R.

7. Evaluating Experimental Results

Experimental data alone tells only part of the story. The ability to properly evaluate results is what transforms raw measurements into meaningful scientific knowledge - a fundamental skill that distinguishes casual observations from rigorous scientific inquiry. This chapter focuses on why this evaluation process is crucial: it validates measurements through uncertainty analysis, connects physical systems with theoretical models, and ultimately determines whether findings represent substantiated scientific knowledge or merely unverified observations.

7.a. The Essential Final Analysis

The primary objective in conducting an experiment is to make substantive statements about relationships between physical systems and theoretical models. This involves:

The evaluation process involves several key analytical steps. First, we must identify patterns and trends in data that either align with or deviate from theoretical predictions. Next, it's essential to quantify the strength of relationships between measured variables to understand how they interact. We'll need to determine whether any observed effects are statistically significant, that is, ensuring findings aren't simply due to random chance. Additionally, accounting for experimental uncertainties and understanding how they impact conclusions is crucial for reliable results. Finally, we must critically assess whether results support, refute, or suggest necessary modifications to existing theoretical models. This comprehensive analysis ensures experimental findings contribute meaningful insights to scientific understanding.

> **Important**
>
> Even after completing all the measurements for an experiment, an equally important phase still remains: evaluating what the results actually mean. This evaluation phase transforms raw data into meaningful scientific conclusions. The analysis stage is where we determine whether experimental results validate or challenge existing theories, and potentially uncover new physical insights.

7.b. Approaching Evaluation with the Right Mindset

> **Important**
>
> Before diving into specific evaluation techniques, two essential principles should guide the approach:
>
> First, recognize that experimental results are precious resources. Whether they come from a multi-million dollar research program or a simple classroom exercise, results represent unique, sometimes irreplaceable information. Honor this by extracting every possible insight from observations and ensuring final conclusions are as complete as possible.
>
> Second, maintain unwavering objectivity. It's nearly impossible to approach an experiment without some preconceptions about what "should" happen. However,

we must discipline ourselves to assess results objectively. If outcomes differ from expectations or hopes, report them honestly and use them constructively to guide future investigations.

Tip

In academic settings, students sometimes misunderstand experimental objectives, believing the goal is to reproduce known values. If a measurement of gravitational acceleration yields 9.60 m/s² instead of the textbook 9.80 m/s², this isn't a failure—it's simply a measurement with its own characteristics and uncertainties. Rather than fretting over differences from established values, focus on making measurements as reliable as possible and accurately assessing their uncertainties.

Note

When measuring quantities with established values, resist comparing results until analysis is complete. This builds confidence in the experimental process—confidence we'll need when eventually measuring previously unmeasured quantities during professional research.

Important

If a gravitational acceleration measurement is 9.60 ± 0.30 m/s², recognize that the uncertainty is as significant as the central value. The 3% uncertainty might reflect equipment limitations rather than experimental shortcomings. Textbook values often appear without context about the sophisticated methods and equipment used to obtain them. The aim should be honest, objective reporting of results with appropriate uncertainty limits—not perfect reproduction of established values in limited laboratory time.

7.c. *The Evaluation Process: Four Essential Stages*

Important

The complete evaluation process consists of four key stages:

1. Calculate the values and uncertainties of basic measured quantities
2. Assess correspondence between experimental system and theoretical model
3. Calculate values of the properties we set out to measure
4. Estimate the overall precision of the experiment

Let's examine each stage in detail.

7.c.i. *Stage 1: Calculating Elementary Quantities:*

Note

> The first task is determining the values and uncertainties of the fundamental quantities involved in the experiment. Consider a pendulum experiment designed to determine gravitational acceleration. We'll likely have measurements of pendulum length (ℓ) as the input variable and time measurements for multiple oscillations as the output, from which we'll calculate oscillation period (T).

The approach depends on whether we're working with estimated uncertainties or statistical treatment of random fluctuations.

Working with Estimated Uncertainties:

> **Tip**
>
> For pendulum length measurements using a meter stick, we've likely determined intervals within which we're confident the true values lie. Results would appear as a set of values in the form: value $\pm$ uncertainty.

Similarly, if we've counted oscillations and timed them with a stopwatch, we might express time measurements with their uncertainty ranges. However, the oscillation period T (the actual variable of interest) must be calculated from these measurements. If we counted 15 oscillations that took 18.4 ± 0.2 seconds, the period for a single oscillation would be:

$(1/15)(18.4 \pm 0.2) = 1.227 \pm 0.013$ seconds

Notice that both the central value and uncertainty must be calculated through this division. This significant modification of uncertainty values is necessary whenever we perform arithmetic operations on basic measurements.

The final result will be a set of ℓ and T values with their associated uncertainties, preparing us for graphical analysis.

Working with Statistical Uncertainties:

> **Important**
>
> If repeated measurements show random fluctuations, we may have collected multiple readings for statistical analysis. We'll need to express these as central values with uncertainties suitable for plotting.

> **Note**
>
> As discussed in earlier chapters, sample means and standard deviations of means provide readily interpretable statistical significance. When reporting results, clearly indicate that we're quoting sample means and standard deviations so readers understand we're specifying intervals with 68% probability of containing the true value.

Warning

Remember that laboratory samples are often too small for definitive assessment of the underlying distribution. We're making an assumption when applying Gaussian distribution properties to the sample, though it's usually reasonable.

Also recall warnings about small-sample statistics—generally, statistical approaches aren't worthwhile with fewer than 10 observations.

Consider how uncertainty regions will be interpreted on the graph. If both variables have similar statistical character, each point's uncertainty rectangle will have clear interpretation. If variables have different uncertainty types (estimated versus statistical), interpretation becomes problematic. We might need to standardize them—perhaps using twice the standard deviation of the mean (95% probability) to make them comparable to estimated uncertainties.

At this stage, every experimental quantity should have a central value and uncertainty, but we're not quite ready for graphing. If we need to plot derived variables (like T^2 vs. ℓ for a pendulum), we must calculate these through arithmetic operations. Remember to properly propagate uncertainties—if plotting T^2 values, uncertainty bars must represent the actual interval over which T^2 is uncertain.

7.c.ii. *Stage 2: Creating Effective Graphs:*

Important

Whether the graph serves as a simple illustration or as the key analytical tool, the goal is displaying results so their characteristics are immediately apparent. This requires thoughtful choices about scale, proportions, and presentation.

Tip

First, ensure graph paper is sufficiently large. Plotting high-precision observations (0.1%) on standard letter-sized paper is futile when graphing uncertainty is around 2%. Unless uncertainties are clearly visible, we'll lose valuable information. Similarly, make the graph fill the available area by choosing appropriate scales and suppressing zero when necessary. When plotting copper wire resistance versus temperature with values ranging from 57-62 ohms, starting the resistance scale at 55 rather than zero creates a meaningful display instead of a "flat roof" over empty graph paper.

There are exceptions where preserving the origin is important—when examining behavior near zero or when illustrating variation relative to baseline values. Generally, however, maximize use of graph space.

Important

Clearly indicate uncertainty ranges for each measurement. We might use crosses with horizontal and vertical bars showing uncertainty ranges, or small rectangles encompassing the measurement with dimensions indicating coordinate uncertainties. The specific method matters less than consistently marking uncertainties on every graph. Note the nature of these uncertainties (estimated limits, standard deviations, etc.) on the graph itself or in its caption to prevent readers from hunting through text for interpretation guidance.

If plotting multiple datasets on one graph, differentiate them clearly through different symbols, colors, or other distinguishing features.

7.c.iii. *Stage 3: Comparing Models with Experimental Data:*

Important

Once observations are plotted, we're ready for the crucial step—comparing system properties with model predictions. The approach varies by circumstance, but we'll assume we've arranged variables for linear graphical representation.

Scenario 1: Fully Specified Model:

Note

If we're working with a completely specified model without undetermined parameters, the goal is simply assessing how well model predictions match experimental observations. Draw the model's function on the same graph as experimental points, using identical scales. This approach was illustrated earlier with falling object observations compared to the theoretical expression:

$$t = 0.4515x^{1/2} \quad (59)$$

How do we judge correspondence quality? This is where uncertainty intervals become crucial. Without them, the inevitable scatter in experimental points would make meaningful comparison impossible—what are the chances of a theoretical line passing exactly through multiple scattered points? When points represent possible value intervals rather than single values, logical assessment becomes possible.

Tip

If the line representing the model passes through each point's uncertainty region (as in the earlier example), we can state this observation directly. This doesn't "prove" the equation is "true" or "correct"—it merely indicates the model and system are "consistent," "in agreement," or "compatible" within our measurement precision. Using appropriate language prevents misrepresentation and potential misunderstanding.

Also recognize that agreement exists only at our current precision level. At higher precision, discrepancies might appear that weren't detectable in the experiment.

Scenario 2: Partial Correspondence:

Note

Sometimes a model adequately describes a system only within certain parameter ranges. The graphical comparison might resemble Figure 6.1(b) or 6.1(c), showing agreement over limited ranges. For instance, fluid flow through a pipe might follow a linear pressure-flow relationship only below turbulence onset, or metal resistivity might follow a linear temperature model except at very low temperatures.

Tip

In such cases, report the comparison using language like: "We observed agreement between model and observations only over the range X to Y" or "The properties diverged significantly beyond value Z."

Important

Resist thinking something is "wrong" when models and systems don't correspond completely. Both exist independently, and we cannot prejudge their overlap extent. Detecting validity limits for particular models often provides valuable clues for model improvement.

Scenario 3: Unexpected Intercepts:

Note

We'll frequently encounter situations where a model's behavior passes through the origin, but experimental observations don't, as shown in Figures 6.1(d) and 6.1(e). Such discrepancies can arise from various model-system mismatches and provide valuable analytical insights.

Tip

When drawing graphs, check behavior at the origin. As previously discussed, graphical analysis helps obtain answers free from systematic errors associated with unexpected intercepts. Knowing whether such intercepts exist helps assess overall correspondence between model and system.

Scenario 4: Unexpected Data Scatter:

Important

During experiment planning, we should have carefully assessed measurement uncertainties and chosen appropriate methods to achieve the target precision. If we've done this properly, the scatter in plotted points should be consistent with estimated measurement uncertainties, as in Figure 6.1(a).

Warning

However, reality often deviates from expectations, and we may find ourselves facing a situation like Figure 6.1(f), where scatter exceeds predicted uncertainty. This usually indicates unforeseen factors in the measurement process that weren't accounted for in the initial uncertainty assessment.

Don't leave such discrepancies unaddressed. Check the apparatus to identify potential fluctuation sources—perhaps a loose electrical connection or unstirred heating bath. Resolving such issues is always satisfying. If continuing the experiment isn't possible, work with existing results and make the best assessment possible of correspondence between model and system, perhaps noting that observations distribute uniformly around the model line.

Scenario 5: Complete Non-correspondence:

Warning

It's rare to encounter situations where system behavior bears no resemblance whatsoever to model behavior [Figure 6.1(g)]. With properly functioning equipment, this outcome is highly unlikely. Models may be imperfect representations of physical reality, but they wouldn't qualify as models if they performed as poorly as this scenario suggests.

Such complete correspondence failure usually indicates experimental error—misinterpreting variables, incorrectly transforming equations, improper equipment setup, or mistakes in observation, calculation, or graphing. If possible, review everything from the beginning. If equipment access isn't possible, check all analytical and arithmetic processes. If all error-finding attempts fail, report results honestly and objectively. We may have discovered something novel, and an honest account of puzzling results from well-checked equipment will interest others in the field.

Important

Throughout this assessment process, remember: experiments don't give “right” or “wrong” results. Our responsibility is conducting experiments carefully and reporting outcomes honestly and objectively. Occasional reminders that models provide only partially satisfactory representations of physical systems are healthy. Understanding model validity limits and failure modes provides invaluable evidence for those seeking to improve them.

7.c.iv. *Stage 4: Determining Values from Straight-Line Analysis:*

Note

In previous sections, we discussed comparing fully specified models (including all numerical values) with experimental systems. However, as explained in earlier chapters, straight-line analysis frequently serves to determine unknown model parameters appropriate for the system.

In these cases, the model contains initially unknown quantities, so we cannot draw a complete model graph for comparison with experimental points. The graph initially contains only the points themselves, as shown in Figure 6.2(a).

Consider measuring current through and potential difference across a resistor to test Ohm's Law (V = IR). Without knowing resistance R, the model behavior encompasses all lines through the origin on the I-V plane described by:

$$V = constant \times I \tag{60}$$

where the constant could be any positive value. In principle, we could draw all possible lines on the graph and determine: (1) the extent to which system and model behaviors overlap, and (2) the range of R values appropriate for the system (as illustrated in Figure 4.11).

In practice, this is complicated by the fact that, based on measurements shown in Figure 6.2(a), we cannot assume system behavior passes through the origin. It's best to defer the intercept question and simply determine which straight lines are consistent with observations.

Finding the "Best" Line and Uncertainty Range:

Important

Several approaches exist for line-fitting. The most rigorous statistical method (least squares) will be discussed later. Meanwhile, we'll examine simpler mechanical procedures, starting with the time-honored practice of drawing the "best" straight line through points by eye.

Tip

This requires mechanical aids that don't obscure half the data points. Avoid opaque rulers; use transparent straight edges or, better yet, dark thread that can be stretched across points and easily repositioned. If visually judging point trends is difficult, hold the graph at eye level and sight along the points—this makes clustering around a straight line or systematic deviations much more apparent than direct viewing.

Identify several significant lines: the "best" straight line by judgment, plus the limiting lines representing how far we can reasonably rotate the "best" line before it no longer acceptably fits the data. These extremes provide uncertainty values for the slope.

If wide point scatter makes identifying best-fit and limiting lines difficult, remember that measured points represent samples from a continuous distribution band. The sparse population of this band (due to limited observations) can complicate line selection. Visualize the band populated by millions of potential readings the apparatus might produce, then estimate the center and edges of that distribution, allowing us to select appropriate lines.

In Figure 6.2(b), we might choose AB as the “best” line and determine that lines CD and EF would contain almost all possible points from an infinite measurement set. Lines CF and ED (not shown) would represent the steepest and shallowest slopes consistent with observations.

Once we’ve selected appropriate lines, determine their slopes numerically to calculate the desired parameter (like resistance R in our Ohm’s Law example). For slope calculation, angle is irrelevant—we need the quantitative relationship between measured variables. For a line like AB in Figure 6.3, identify precise coordinates where it crosses graph grid intersections near its endpoints. If these coordinates are (I_1, V_1) and (I_2, V_2), calculate:

$$slope = (V_2 - V_1)/(I_2 - I_1) \tag{61}$$

For our example, R equals this slope directly. In more complex cases, we might need additional calculations involving other measured quantities to determine the final answer.

Perform this process three times: once for the “best” line (AB) and once each for upper and lower limiting lines (CF and ED). This gives the best value for R plus upper and lower limits beyond which we’re “almost certain” the true value doesn’t lie. Typically, these extreme values are roughly equidistant from the central value, allowing us to express the result as:

$$R = value \pm uncertainty \tag{62}$$

Sometimes the “best” line and limiting lines won’t appear equally spaced, usually because too few points prevent good line assessment. While sometimes experimenters feel compelled to express asymmetric uncertainties as:

$$value(+uncertainty_1 / -uncertainty_2) \tag{63}$$

visual graph judgment rarely justifies such precision. If identifying a clear “best” line proves genuinely difficult, we can simply delineate the edges of the value band (lines ED and CF in Figure 6.3), calculate maximum and minimum slopes, and express the experimental result as the interval between these slopes, or as their average $\pm$ half their difference.

If the desired answer isn't directly equal to the slope, but requires calculation using additional quantities with their own uncertainties, combine the slope uncertainty with these other uncertainties using methods described in Chapter 2.

The significance of uncertainty values obtained from graphs depends on how we marked uncertainty on original data points. If the bars represented outer limits of possible variation (either subjectively assessed or $2S_m$ for statistical fluctuations), the slope limits have similar interpretation. If points were marked with $1S_m$ limits, the limiting slopes probably represent better than 68% probability because of the conservative approach used in drawing limiting lines.

This analysis assumes that actual data scatter falls within predicted uncertainty ranges. If scatter greatly exceeds expected uncertainty (due to unforeseen fluctuation sources), we may have difficulty establishing lines that contain "almost all" possible values with confidence. In such cases and for all precision work, least squares analysis (discussed later) becomes essential.

When selecting the three lines, deliberately exclude the origin from consideration, as system behavior at the origin may be one aspect we wish to examine. If the model should pass through the origin, check whether the area between limiting lines includes the origin. If so, the model and system show consistency at our precision level. Only if both limiting lines clearly intersect an axis on the same side of origin can we confidently identify an unexpected intercept.

If the model predicts an intercept from which we hope to determine some quantity, the intersection of the three lines with the relevant axis directly provides that intercept as: value $\pm$ uncertainty.

7.d. Handling Imperfect Model-System Correspondence

> **Important**
>
> When model and system correspond only partially, exercise care to avoid introducing systematic errors from these discrepancies into results. Consider first cases where measurements align with the model's straight line only over limited ranges [Figures 6.1(b) and 6.1(c)].

Obviously, restrict slope evaluations to regions where system and model are compatible. Points systematically deviating from the straight line reflect physical circumstances not included in the model, making them inappropriate for model-based calculations. Disregard all points deviating systematically from straight-line behavior by amounts clearly exceeding estimated uncertainties and observed scatter, limiting slope and uncertainty calculations to the linear region.

> **Tip**
>
> A second consideration involves intercepts. Even when the model passes through the origin, graphs frequently show intercepts. Such deviations arise from various

causes, but many prove harmless. If the intercept-causing discrepancy affects all readings equally (like undetected zero error in an instrument or constant spurious EMF in an electrical circuit), the graph's slope remains free from the systematic error that would otherwise contaminate results.

Therefore, design experiments so answers come from graph slopes, while quantities potentially subject to undetermined systematic errors appear as intercepts. This capability to provide answers free from many systematic error types represents one of graphical analysis's principal advantages.

7.e. The Principle of Least Squares

Important

All previously described procedures share a common limitation—they rely on experimenter judgment. While these approaches are useful and common, they invite criticism that even when carefully applied, their numerical significance remains uncertain. Having a mathematical procedure to identify the "best" line for a dataset would free us from judgment-related insecurity and potentially provide insight into what "best" actually means while allowing more precise uncertainty calculation.

The method meeting these needs is based on the statistical principle of least squares. We'll focus primarily on its application to straight-line fitting, though it can be extended to other functions.

Note

Consider a set of N (x,y) measurement pairs where uncertainty is confined to the y-dimension—we'll assume x values are exactly known or sufficiently more precise than y values that x-dimension uncertainty can be neglected. This assumption is reasonable for many experimental situations where one variable is significantly more precise than the other. If both variables have comparable uncertainty, more complex treatments are needed (see Wilson's text in the Bibliography).

Our mathematical procedure must answer: Which line on the x-y plane is "best," and what does "best" mean? Least squares makes this determination based on vertical deviations of points from a candidate line. For line AB in Figure 6.4, consider vertical intervals between points and line (like P_1Q_1 and P_2Q_2). The "best" line minimizes the sum of squares of these deviations.

This criterion offers no automatic path to "truth" or "correct" answers—it's simply one optimization criterion among many possibilities (we could minimize third powers or first powers of intervals, etc.). However, it can be proven that minimizing squared deviations produces smaller variance in resulting parameters (like slope) upon repeated sampling than any alternative criterion. This provides greater confidence in least squares results than competing approaches, explaining its near-universal adoption.

Mathematically, we define the best line as that which minimizes:

$$\sum_i (P_i Q_i)^2 \tag{64}$$

giving parameters (slope m and intercept b) for that line.

If our line equation is $y = mx + b$, each deviation δy_i equals the difference between measured y value and the corresponding point on the line:

$$\delta y_i = y_i - (mx_i + b) \tag{65}$$

The least squares criterion seeks to minimize:

$$\sum_i [y_i - (mx_i + b)]^2 = \chi \tag{66}$$

with conditions:

$\frac{\partial M}{\partial m} = 0$ and $\frac{\partial M}{\partial b} = 0$

Solving these equations (full derivation in Appendix A2) yields formulas for the best-fit line parameters:

$$m = \frac{N \sum (x_i y_i) - \sum x_i \sum y_i}{N \sum x_i^2 - \left(\sum x_i\right)^2} \tag{67}$$

$$b = \frac{\sum x_i^2 \sum y_i - \sum x_i \sum (x_i y_i)}{N \sum x_i^2 - \left(\sum x_i\right)^2} \tag{68}$$

We've now replaced potentially questionable visual judgment with a mathematical procedure yielding results of well-defined significance and universal acceptability. Since this method has statistical foundations, we can expect more precise uncertainty calculations. The least squares principle immediately provides standard deviations for slope and intercept, giving uncertainties with known statistical significance.

These standard deviations are calculated using the standard deviation of y-value deviations from the best line, Sy:

$$S_y = \sqrt{\frac{\sum (\delta y_i)^2}{N - 2}} \tag{69}$$

Don't worry about the N-2 denominator rather than the familiar N or N-1; it results from applying standard deviation definition to line positioning on a plane. The standard deviations for slope and intercept are:

$$S_m = S_y \sqrt{\frac{N}{N \sum x_i^2 - \left(\sum x_i\right)^2}} \tag{70}$$

$$S_b = S_y \sqrt{\frac{\sum x_i^2}{N \sum x_i^2 - \left(\sum x_i\right)^2}} \tag{71}$$

Full derivations appear in Appendix A2.

These standard deviations, combined with m and b values, determine intervals with normal statistical interpretation—one standard deviation gives 68% probability of containing the true value, two standard deviations 95%, etc. A key least squares advantage is providing statistically significant uncertainty values for slope and intercept. These values derive objectively from actual point scatter, independent of any optimistic claims about measurement precision.

Appendix A2 also describes an extension for unequally precise data points, allowing greater weight for more precise measurements. This "weighting" procedure applies whenever we combine observations of unequal precision, even for simple tasks like finding the mean of unequal-precision values. Weighted mean and weighted least-squares calculation formulas appear in Appendix A2.

7.f. Least-Squares Fitting for Nonlinear Functions

> **Note**
>
> The procedures used for determining best-fit straight line parameters can, in principle, apply to nonlinear functions. We can write analogous deviation equations for any function and use similar minimization requirements for parameters in our chosen model. If resulting equations are solvable, we can find parameter values as we did for straight lines.

Frequently, however, these equations resist straightforward solution. In such cases, we abandon analytical approaches in favor of iterative computer solutions. We construct trial functions, calculate squared-difference sums, and progressively vary function parameters until finding the minimum sum. Computer-based methods for this process are described in Draper and Smith's text (Bibliography). When possible, testing models in linear form remains simpler.

In all cases, experimenters are responsible for choosing appropriate functions—least squares merely determines which parameter values within a chosen function class best fit the observations.

7.g. Important Cautions When Using Least Squares

> **Warning**
>
> Least squares mathematical procedures are entirely objective and impartial. Equations for linear fitting will drive a straight line through any dataset, regardless of whether a straight-line function is appropriate. If the experiment produces observations clearly showing breakdown of a linear model (Figure 6.5), blindly applying least squares to all observations will yield parameters for line AB

> that have no significance for either model or system. Thoughtless application of least squares methods must be scrupulously avoided.

> **Important**
>
> This warning is particularly important given easy access to calculators and computers that generate least squares parameters for any inputted numbers with a few button presses. Remember that we compare straight lines with observations because we've judged this comparison reasonable. Therefore, never use least squares procedures before plotting observations and visually confirming linear fitting's appropriateness. As mentioned earlier, we may need to exclude observations outside the model's scope from best-line determination.

Only after carefully considering the entire situation graphically and visually, and confirming linear fitting's appropriateness over all or part of the observation range, are we justified in applying least squares. Ignoring this warning can cause serious experiment interpretation errors.

7.h. Finding Functions When No Model Exists

> **Note**
>
> Our previous discussion assumed we possessed a model to compare with the system. While this is common, sometimes we encounter observation sets with no available model—perhaps when researching previously unobserved phenomena or studying systems too complex for theoretical modeling. When plotted, such observations typically show curves with no readily identifiable pattern. Without a model, what approaches are available?

One option is finding functions with some correspondence to observations. This can be valuable in complex systems where theoretical modeling seems hopeless. Even if the "model" is merely a mathematical function restating the system's behavior, it facilitates computer processing and enables interpolation, extrapolation, and similar operations. Such empirical models help predict national economic responses to taxation changes or determine temperatures from resistance thermometer calibration curves.

> **Warning**
>
> In simpler systems where theoretical modeling seems possible, functions showing good correspondence with observations may guide model building by suggesting underlying physical processes. However, caution is essential. Finding a function consistent with observations at a particular precision level doesn't "prove" we've discovered the "right" function. Different function types often show similar behavior over limited variable ranges, and guidance from incorrectly identified functions can be misleading, potentially impeding theoretical progress for years.

> The physics history contains many examples where researchers failed to recognize empirical function choices must remain provisional.

With appropriate caution regarding potential limitations, here are some common function-finding approaches:

7.h.i. *Power Law Functions:*

Tip

As discussed in experiment planning chapters, logarithmic plotting helps identify power law relationships. Consider the function:

$$y = x^a \tag{72}$$

Taking logarithms of both sides:

$$logy = a \cdot logx \tag{73}$$

A graph of log y versus log x produces a straight line with slope a. To test whether observations follow a power law, plot them as log y versus log x. If points align with a straight line, we can conclude a simple power function (positive/negative, integral/fractional as determined by the graph) fits observations well. The appropriate power value comes from the graph's slope, with uncertainty limits depending on plotted point uncertainties.

Such graphs can use ordinary paper (plotting actual log x and log y values) or logarithmic graph paper (with rulings proportional to logarithms, allowing direct plotting of original values).

7.h.ii. *Exponential Functions:*

Tip

Many physical phenomena follow exponential relationships:

$$y = ae^{bx} \tag{74}$$

Taking natural logarithms:

$$\ln y = \ln a + bx \tag{75}$$

This creates a straight line when plotting ln y versus x (a "semi-log plot"). If we suspect an exponential function might apply to the system, create a semi-log plot using either ordinary graph paper (calculating ln y values) or semi-log paper (with one logarithmic and one linear scale). Appropriate a and b values come from the intercept and slope, with uncertainties determined by measurement precision.

7.h.iii. *Polynomial Approximations:*

Note

> If neither power laws nor exponential functions adequately match observations, the chances of finding a more complex function that fits well are slim. In such cases, polynomial approximations often prove useful:
>
> $$y = a_0 + a_1 x + a_2 x^2 + \cdots \tag{76}$$
>
> While such representations essentially admit ignorance about underlying system mechanisms, they still offer empirical modeling advantages—facilitating computation and providing bases for interpolation and extrapolation.

Finding appropriate coefficients for such expansions typically employs the least squares principle. As noted earlier, computational difficulty increases rapidly with the number of terms needed for satisfactory correspondence. Fuller discussion appears in Draper and Smith's text (Bibliography).

Similar approaches apply when observation scatter isn't severe and highest precision isn't essential. Finite difference calculus techniques can be applied to observations, and difference tables used for interpolation, extrapolation, or polynomial fitting. Comprehensive discussion appears in texts by Whittaker and Robinson and by Hornbeck (Bibliography), with elementary treatment in Appendix A3.

7.i. *Assessing Overall Experimental Precision*

> **Important**
>
> At experiment initiation, we estimated likely uncertainties to guide the experimental approach. After completion, we should retrospectively evaluate actually achieved precision through critical results assessment. The specific uncertainty type matters less than clearly stating what we're reporting—whether estimated ranges, standard deviations, standard deviations of means, or other measures.

> **Warning**
>
> For meaningful application, overall uncertainty figures must be realistic and honest, even when experimental outcomes are less favorable than hoped. Include all identifiable uncertainty sources in the assessment. If balance points cannot be identified within 2-3 mm or slide wire non-uniformities introduce errors, claiming 0.2% precision for slide-wire potentiometer readings becomes meaningless, regardless of millimeter-graduated scales.

Known systematic error contributions should be excluded at this stage, as appropriate measurement corrections should already have been applied. However, suspected systematic error sources whose contributions cannot be accurately evaluated should be described with appropriate allowances in overall uncertainty ranges. Final statement format depends on circumstances:

7.i.i. *For Results Based on Measurement Sets:*

Tip

The best quantity to report is standard deviation of the mean, which has recognizable numerical significance. Sometimes standard deviation itself is quoted. Always specify sample size so σ estimate reliability can be judged.

7.i.ii. *For Results from Single Calculations:*

If graphical analysis wasn't possible and results come algebraically from several measured quantities, use Chapter 3 methods to calculate either outer uncertainty limits or standard deviations.

7.i.iii. *For Results from Graphical Analysis:*

Note

If the straight line was established through least squares, constant uncertainties (m and b) will have been directly determined. These uncertainties advantageously derive from actual point scatter, independent of estimated uncertainties. (This doesn't mean we should skip plotting uncertainties or drawing graphs when using least squares—as emphasized earlier, graphs with plotted uncertainties remain essential for judging model-system correspondence ranges before least squares calculations).

If we've drawn the line by eye, the limiting possibility lines will give slope and intercept ranges. This slope uncertainty may need combining with other quantity uncertainties before stating final answer uncertainty.

Tip

As mentioned previously, the specific uncertainty type matters less than clearly stating what we're reporting. When working through lengthy uncertainty calculations, we can simplify by dropping insignificant contributions—adding 0.01% uncertainty to 5% offers negligible benefit since the 5% value lacks three-digit precision. Final uncertainty statements rarely justify more than two significant figures; only highly significant statistical work warrants greater precision.

Once we've determined overall answer uncertainty, consider how many significant figures to retain. This was covered in Section 2.11, but bears repeating in the context of experiment evaluation.

Important

No unique answer exists for significant figure questions, but generally avoid retaining figures beyond the first uncertain digit. For example, 5.4387 ± 0.2 should be reported as 5.4 ± 0.2, since uncertainty in the tenths position makes

> subsequent digits meaningless. If uncertainty is known more precisely, retaining one additional figure might be justified—if uncertainty were 0.15, reporting 5.44 ± 0.15 would be valid.

When reporting percentage precision, significant figures are automatically implied. A measurement reported as $527.64182 \pm 1\%$ implies absolute uncertainty of 5.2764. However, since precision is quoted to just one significant figure (1%, not 1.000%), the uncertainty itself warrants only one significant figure. Calling it 5 implies the tens digit in the original number is uncertain by 5, making subsequent digits meaningless. The measurement should be quoted as 528 ± 5 or $528 \pm 1\%$.

For sample means, significant figures depend on the mean's standard deviation, which in turn depends on the standard deviation's standard deviation.

Finally, always ensure answer and uncertainty expressions are consistent—neither "16.2485 ± 0.5" nor "4.3 ± 0.0002" represents good practice.

7.j. Understanding Correlation

> **Note**
>
> Thus far, we've considered experimental interpretation involving relatively precise observations and satisfactory models. Reality is often messier, and much modern experimentation is less clear-cut than previous sections might suggest.

Many scientific fields deal with subtle phenomena where effects can be partially or completely masked by statistical fluctuations or other perturbations. In these scenarios, detailed model-system comparisons may be impossible—we might struggle even to establish whether the effect we're studying exists at all. This scenario commonly occurs in biological, medical, and environmental studies.

Consider familiar public health debates about smoking's role in lung cancer, low-level radiation's relationship to leukemia, or dietary influences on cardiovascular disease. In these contexts, "proof" frequently enters discussion: "We haven't proved smoking causes lung cancer" or "Can we prove heart attacks are less frequent with margarine versus butter consumption?"

These scenarios operate in fundamentally different domains from our earlier experimental approaches. Understanding what we mean by terms like "proof" and "cause" becomes critical.

> **Important**
>
> Consider two experimental scenarios. First, measuring current through a resistor as potential difference varies, producing results like Figure 6.6(a). Have we "proved" current is "caused" by potential difference? The current certainly differs between low and high potential differences by amounts far exceeding measurement uncertainty, giving confidence the variation is real. Was this

> variation “caused” by potential difference changes? We observed current increases with potential difference increases, but theoretically, current might be unrelated to potential difference, with increases caused by some entirely separate factor like atmospheric pressure, making the apparent relationship purely coincidental.

Philosophers have warned for centuries that simultaneous events aren’t necessarily causally related. However, accumulated experience with this experiment, involving multiple repetitions and careful control of other variables, gradually convinces us potential difference and current are genuinely related. Only philosophical purists would dispute that potential difference causes current flow.

The situation differs dramatically in less clear-cut cases. Another experiment might yield results like Figure 6.6(b), typical when studying, for instance, university student cold incidence versus daily vitamin C consumption. Can we conclude cold frequency depends on vitamin C dosage? We might conduct a well-designed experiment with 100 students receiving vitamin C supplements versus a control group receiving placebos, but multiple confounding factors could mask any real effect—or create apparent effects where none exist.

7.k. Glossary

7.l. Problems

Exercise 54:

A student measures the voltage across a resistor at different currents and obtains the following data:

Current (mA)	Voltage (V)
10 ± 1	2.1 ± 0.1
20 ± 1	4.0 ± 0.1
30 ± 1	6.2 ± 0.1
40 ± 1	8.1 ± 0.1
50 ± 1	10.0 ± 0.1

Draw a best-fit line through the data and determine the resistance with its uncertainty using the graphical method described in this chapter.

Exercise 55:

Using the data from the previous problem, apply the least squares method to find: a. The best-fit slope and intercept b. The standard deviations of the slope and intercept c. Compare your results with those obtained from the graphical method.

Exercise 56:

An experiment investigating the relationship between pendulum period T and length L yields the following results:

Length (cm)	Period (s)
20	0.91
40	1.28
60	1.54
80	1.78
100	1.99

The theoretical model predicts $T = 2\pi\sqrt{L/g}$. Assess the correspondence between this model and the experimental data, and determine a value for g with appropriate uncertainty.

Exercise 57:

A student plots voltage versus current for a circuit element and obtains a straight line with a non-zero intercept on the voltage axis. List three possible physical causes for this unexpected intercept and explain how each would affect the interpretation of the results.

Exercise 58:

The following data relates the intensity of light at various distances from a point source:

Distance (m)	Intensity (W/m²)
1.0	100
1.5	44
2.0	25
2.5	16
3.0	11

Use a logarithmic plot to determine if this data follows a power law relationship $I = kd^n$, and find the values of k and n.

Exercise 59:

A radioactive sample produces the following count rates at different times:

Time (min)	Count rate (counts/s)
0	1000
5	607
10	368
15	223
20	135

Use a semi-log plot to verify the exponential decay model and determine the half-life of the sample.

Exercise 60:

When plotting experimental data, you observe that the scatter of points about the best-fit line is significantly larger than expected from your estimated measurement uncertainties. Describe three approaches you might take to address this situation and explain the advantages and limitations of each.

Exercise 61:

A study finds a strong correlation between ice cream sales and drowning incidents. Explain why correlation does not imply causation in this case, and describe what additional information or experimental design would be needed to establish a causal relationship.

8. Writing Scientific Reports

8.a. Why Quality Scientific Writing Matters

Important

The value of excellent scientific writing cannot be overstated. Even groundbreaking experimental work loses its impact if poorly communicated. While verbal presentations occasionally suffice, most scientific communication happens through written documents. Developing strong writing skills should be considered a fundamental component of the experimental toolkit.

Note

Learning to write well cannot be reduced to a simple checklist. Each person develops their own distinctive writing style through practice. The introductory physics laboratory provides an excellent opportunity to develop these skills. Our writing styles may differ, but clarity remains the essential common element that makes diverse approaches valuable rather than problematic.

When approaching scientific report writing, one guiding principle stands above all others: focus on the reader. Whether preparing an internal technical document or a manuscript for publication, prioritize the needs of the person who will read the work. From their perspective, we are communicating across distance and time, with only written words to convey the message. We cannot clarify misunderstandings or add explanations as they read. The report must stand independently and communicate effectively on first reading. The reception of the work, its scientific impact, and potentially professional advancement may hinge on how effectively readers understand the report during their brief engagement with it. This perspective should emphasize the importance of taking writing seriously.

Contemporary scientific writing has largely moved away from impersonal, passive constructions. Instead, straightforward language often works best: "We measured the fall time using a millisecond-precision electronic timer." With no single "correct" approach to report writing, choose language that communicates most clearly and engagingly. For additional guidance on effective writing, consider consulting Strunk and White's classic text referenced in the Bibliography.

Let's examine each report section from the reader's perspective.

8.b. Title

Important

The title typically provides readers their first impression of the work. Since potential readers are usually busy professionals with competing demands on their attention, the title must be informative, appropriate, and engaging.

Tip

> While keeping it concise, make the topic explicit. For instance, if measuring a fluid's specific heat using continuous-flow calorimetry, a straightforward title works well: "Measurement of the Specific Heat of Water Using Continuous-Flow Calorimetry." This title effectively answers three key questions:
>
> 1. Is this experimental or theoretical work?
> 2. What specific topic does it address?
> 3. What methodology was employed?

Addressing these elements typically results in an effective title.

8.c. Format

> **Note**
>
> The following sections analyze report components in detail. Note that the subsection headers discussed here should not appear in the actual report. Most standard physics laboratory reports require only minimal sectioning.

> **Important**
>
> Essential sections typically include:
>
> INTRODUCTION
> PROCEDURE
> RESULTS
> DISCUSSION
>
> These core divisions provide a solid organizational framework. Format headings prominently, perhaps using capital letters. Use subsections sparingly, only when necessary for clarity in longer or more complex reports. Depending on experimental requirements, we might include additional main sections such as:
>
> THEORY SAMPLE PREPARATION UNCERTAINTY CALCULATIONS
>
> The report should present a clear, logical progression of ideas. If detailed information might disrupt this flow, consider placing it in an appendix. This approach preserves the information for interested readers while maintaining the report's coherence.

Now let's examine each section's specific requirements.

8.d. Introduction

> **Important**
>
> An effective introduction typically includes these components in sequence:
>
> - Topic Statement
> - Review of Existing Information

- Application of Information to Specific Experiment
- Summary of Experimental Intention

8.d.i. *Topic Statement:*

With a well-crafted title having captured reader attention, remember that readers likely begin with minimal knowledge of the specific experiment. Rather than immediately diving into experimental details, begin with a broad framing statement. For example: "It is possible to measure gravitational acceleration by observing a simple pendulum's oscillation." This approach guides readers from their initial unfamiliarity to a clear understanding of the work's focus.

8.d.ii. *Review of Existing Information:*

At this point, readers need contextual background. Provide a concise summary of relevant knowledge, potentially including historical context or previous experimental findings. Two elements must appear in every experimental report: a clear description of the system and experimental conditions, and an explanation of the theoretical model(s) employed.

Keep this background information concise to maintain focus on the main argument, while ensuring readers have sufficient context to understand the work. Standard theoretical derivations should be omitted, but the resulting equations and their limitations should be included. For example: "It can be demonstrated that for vanishingly small oscillation amplitudes, a simple pendulum (modeled as a point mass on a massless, inextensible string) has a period given by…". If readers might need derivation details, cite an appropriate reference.

8.d.iii. *Application of Information to Specific Experiment:*

Having established context, readers will wonder how this background relates to the specific experiment. Explain how general principles apply to the particular work. This often involves showing how theoretical equations can be transformed into a useful experimental framework, such as rearranging equations into straight-line form for graphical analysis. Identify how system-model comparisons will be made and what information can be extracted from parameters like slopes and intercepts. This preparation ensures readers understand how the final results will be obtained.

8.d.iv. *Summary of Experimental Intention:*

Conclude the introduction by summarizing the specific experimental goals. For example: "Thus, by measuring refractive index variation with wavelength, we can test Cauchy's model using n versus $1/\lambda^2$ graphical analysis. The Cauchy coefficients A and B for our glass specimen will be determined from the graph's intercept and slope, respectively." This summary provides readers with a framework for understanding the experimental procedure that follows, particularly valuable in complex reports with extended introductions.

8.d.v. *Statement of Experimental Purpose:*

While not mentioned above, include a clear purpose statement somewhere in the introduction. Its placement depends on context. For familiar topics, it can effectively serve as the opening topic statement: "The purpose of this experiment is to measure gravitational acceleration by timing a freely falling object." For complex topics, the purpose statement might better follow explanatory material: "…and the purpose of this experiment is to determine coefficient k in equation 10." The statement's placement is flexible, provided it appears where readers can readily comprehend it.

A well-crafted introduction accomplishes several goals: directing reader attention to the research area, providing necessary background, explaining how this context applies to the specific work, and clearly stating experimental objectives. This prepares readers for understanding the experimental process.

8.e. Procedure

> **Tip**
>
> Like the introduction, the procedure section should progress from general to specific. Diving immediately into technical details would confuse readers who lack an overview of the approach. Maintain the same consideration for reader comprehension here as in the introduction, again moving from broader concepts to specific details.

8.e.i. *Outline of Procedure:*

Begin by providing an overview of the experimental approach. If the experiment measured copper wire resistance variation with temperature between 20°C and 100°C, state this clearly to establish a framework for subsequent details. Starting instead with specific connections and instrument readings would quickly lose reader attention without this contextual foundation.

8.e.ii. *Specific Measurement Details:*

With the general experimental approach established, readers can now appreciate specific measurement methodologies. Systematically describe how each required quantity was measured. Ensure no significant measurement approach is omitted; readers need to know whether we used a millisecond-precision electronic timer or a 0.2-second resolution stopwatch. Standard techniques may require only brief mention: "Resistances were measured using a Wheatstone bridge accurate to 0.01%." We might discuss measurement accuracy here, while reserving discussion of overall experimental precision for later sections.

8.e.iii. *Precautions:*

After describing measurement methods, readers may wonder about potential errors inherent in these techniques. Assure them that we anticipated these issues and took appropriate precautions. Exercise judgment here—routine precautions need not be elaborated, but special measures taken to address significant error sources deserve mention before concluding this section.

8.e.iv. *Apparatus Diagrams:*

Well-executed apparatus diagrams substantially enhance report quality. While professional publications require polished illustrations, even student reports benefit from careful diagramming. Use straightforward tools like rulers to ensure neatness, with clear labels to aid comprehension. Good diagrams save considerable explanatory text and provide details that would be tedious to describe verbally. Reference diagrams at appropriate points in the text; an overview diagram works particularly well when introducing the procedure section: "Using the apparatus shown in Figure 1, we measured ball bearing fall times over heights ranging from 20 cm to 150 cm." Figure 1 demonstrates an acceptable apparatus diagram.

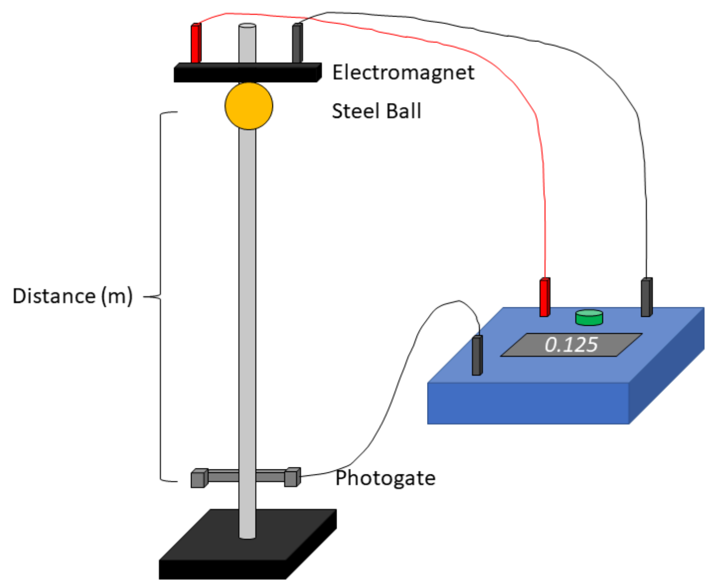

Figure 11: A free-fall apparatus diagram illustrating the experimental setup for measuring the fall times of objects dropped from various heights.

8.f. Results

Important

By now, readers understand the experimental context and methodology, and are ready for the findings. Since most valuable experiments examine relationships between variables, results often benefit from tabular presentation, particularly when not directly comparing to mathematical models.

Tip

Maintain high standards of clarity in tables, with comprehensive headers including variable names, symbols, and measurement units. Include uncertainty

> values with measurements unless addressed separately. Properly identify tables with numbers and titles.

8.f.i. *Measured Values:*

By now, readers understand the experimental context and methodology, and are ready for the findings. Since most valuable experiments examine relationships between variables, results often benefit from tabular presentation, particularly when not directly comparing to mathematical models. Maintain high standards of clarity in tables, with comprehensive headers including variable names, symbols, and measurement units. Include uncertainty values with measurements unless addressed separately. Properly identify tables with numbers and titles. Reference any graphical representations straightforwardly: “Figure 2 shows time of fall versus height.” Place exceptionally detailed data tables in appendices to maintain narrative flow. Following primary results tables, list all other relevant measured quantities with their uncertainties and units.

8.f.ii. *Description of Measurement Uncertainties:*

Clearly specify what kind of uncertainties the values represent—whether estimated maximum limits or statistical measures like standard deviations. For statistical values, include the sample size. When reporting calculated quantities derived from measurements, explain the uncertainty calculation method without necessarily including arithmetic details, provided the approach is clear.

8.f.iii. *Computation of Final Answer:*

Well-designed experiments typically yield final results through graphical analysis. Explain the analytical approach explicitly. Even for straightforward analyses, be specific: “We determined resistance from the slope of the V versus I graph (Figure 3) between 0.5A and 1.5A.” If the result required additional calculations beyond graphical values, state this clearly: “We calculated the viscosity coefficient using the Q versus P graph slope combined with measured values of a and ℓ according to Equation (3).”

Similarly, explain the approach to uncertainty calculation. Whether we visually estimated slope ranges, combined multiple uncertainty sources, or employed least-squares methods with statistical analysis, describe the approach without burdening readers with extensive calculations. If detailed calculations seem necessary, place them in an appendix where interested readers can access them without disrupting the main narrative.

8.g. Graphs

> **Important**
>
> The graphs created during analysis served as computational tools, potentially requiring large formats and precise drawing for accurate measurement. However,

> graphs included in the report serve a different purpose—they illustrate results rather than providing readers raw analytical material. They help readers visualize system behavior and evaluate interpretations.

> **Tip**
>
> Report graphs should be clean, clear, and uncluttered. Plot points with visible uncertainty indicators (boxes or crosses) and clearly label axes. Identify uncertainty types and axis symbols directly on or near the graph to avoid forcing readers to search the text for interpretation.

The graphs created during analysis served as computational tools, potentially requiring large formats and precise drawing for accurate measurement. However, graphs included in the report serve a different purpose—they illustrate results rather than providing readers raw analytical material. They help readers visualize system behavior and evaluate interpretations.

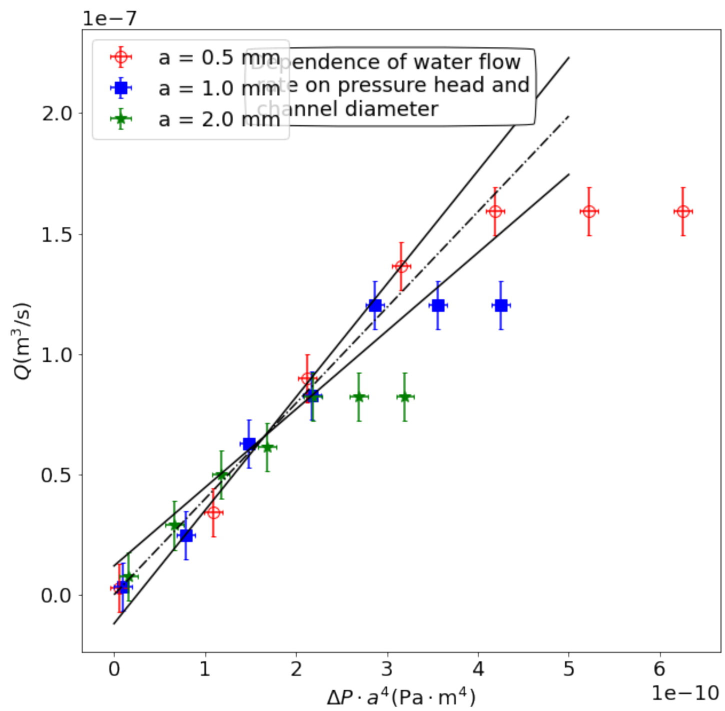

Figure 12: An example of a well-formatted results graph with clearly labeled axes, visible uncertainty indicators (error bars), and a properly identified best-fit line.

Report graphs should be clean, clear, and uncluttered. Plot points with visible uncertainty indicators (boxes or crosses) and clearly label axes. Identify uncertainty types and axis symbols directly on or near the graph to avoid forcing readers to search the text for interpretation. Avoid cluttering graphs with calculation details. Each graph should have a descriptive title or extended caption that can also incorporate important technical details. Figure 2 demonstrates acceptable graph formatting.

8.h. Discussion

> **Important**
>
> The discussion constitutes an integral report component rather than an afterthought. Here we address the fundamental experimental question—the relationship between system and model. This comparison's outcome represents a critical experimental result that readers eagerly anticipate.

> **Tip**
>
> Having evaluated results objectively during analysis, present an unbiased assessment of system-model correspondence. Make a straightforward factual statement about what we observed.

8.h.i. *Comparison Between Model and System:*

The discussion constitutes an integral report component rather than an afterthought. Here we address the fundamental experimental question—the relationship between system and model. This comparison's outcome represents a critical experimental result that readers eagerly anticipate.

Having evaluated results objectively during analysis, present an unbiased assessment of system-model correspondence. Make a straightforward factual statement about what we observed. For example: "The model described by Equation (1) predicts a linear Q versus P relationship passing through the origin. Our experimental results show linear behavior through part of the range, but with a non-zero Q-axis intercept. Additionally, at higher P values, measurements systematically deviate from linearity beyond measurement uncertainty."

Begin with this objective assessment before proceeding to interpretation. Since system-model comparison represents the experiment's fundamental purpose, clearly state the actual results before introducing subjective elements.

This factual statement will naturally raise questions that require attention.

8.h.ii. *Consequences of Discrepancies Between Model and System:*

Readers will wonder whether model-system discrepancies affected the final results. Address how we protected conclusions from potential errors arising from these discrepancies. Explain, for instance, how an unexpected intercept didn't compromise results derived solely from slope analysis, or how nonlinearity didn't affect conclusions drawn from the linear region. Demonstrate how the experimental approach safeguarded results against such complications.

8.h.iii. *Speculation Concerning Discrepancies Between System and Model:*

Earlier report sections emphasized objective reporting of observations and processes. Now, however, we should introduce thoughtful interpretation. Having informed readers about system-model correspondence and protected results from discrepancy-related errors, we've fulfilled core experimental responsibilities. However, readers will naturally be curious about any observed discrepancies. Since we selected a model expected to match the system closely, discrepancies warrant explanation. As the people most familiar with the experiment, we're uniquely positioned to interpret unexpected observations.

Some discrepancies have readily identifiable causes. Flow rate measurements deviating from linearity at high pressure differences might confidently be attributed to turbulence onset. If detecting turbulence was an experimental objective, this observation fulfills the purpose. In other cases, more extensive interpretation may be needed. If measuring viscosity was the primary goal, readers might question why we didn't avoid conditions where laminar flow theory fails. Perhaps turbulence appeared at unexpectedly low pressures; candidly acknowledge this and consider potential causes.

When confronting genuinely puzzling discrepancies, speculation remains valuable even if limited. Our insights, even tentative ones, likely benefit other researchers given direct experimental experience. Conversely, if we cannot offer constructive ideas despite careful analysis, honest acknowledgment of unresolved discrepancies between well-established systems and models can itself contribute meaningfully to scientific discourse.

When speculating about discrepancies, maintain scientific responsibility. Rather than offering disconnected hypotheses, develop interpretations logically connected to observed patterns. For instance: "The T^2 versus m plot's non-zero intercept at m=0 suggests the presence of additional unaccounted mass in our system." Identifying the specific source is less critical than recognizing the logical implications of observations. Such reasoned inference facilitates further investigation by providing a structured framework for subsequent research.

8.i. Glossary

8.j. Problems

Exercise 62:

For each of the following experiments, write an appropriate title that addresses the three key questions: (1) Is this experimental or theoretical work? (2) What specific topic does it address? (3) What methodology was employed?

a. Measuring the speed of sound in air using resonance tubes b. Investigating how temperature affects the resistance of a copper wire c. Determining the focal length of a converging lens using object-image distances

Exercise 63:

A student writes the following introduction for a pendulum experiment:

“We measured the period of a pendulum at different lengths. The equation is $T = 2\pi\sqrt{L/g}$. We got good results.”

Identify the deficiencies in this introduction and rewrite it to include all essential components: topic statement, review of existing information, application to the specific experiment, and summary of experimental intention.

Exercise 64:

The following procedure description appears in a lab report:

“We set up the circuit and measured the voltage and current. Then we changed the resistance and repeated.”

Explain why this description is inadequate for a scientific report and describe what additional information should be included for reproducibility.

Exercise 65:

A student presents a graph with the following characteristics:

- Axis labels show only “V” and “I” without units
- Data points are shown but without error bars
- The scale starts at zero for both axes, leaving most of the graph area empty
- No title or caption is provided

List the improvements needed to make this an acceptable scientific graph and explain why each change is important.

Exercise 66:

After measuring the acceleration due to gravity and obtaining $g = 9.60 \pm 0.15$ m/s^2, a student writes in the discussion:

“Our result of 9.60 m/s^2 is wrong because it should be 9.81 m/s^2. The experiment failed due to errors.”

Critique this discussion and rewrite it to demonstrate proper objective analysis of experimental results.

Exercise 67:

Describe the essential elements that should be included in an apparatus diagram for an experiment measuring the specific heat of a metal using the method of mixtures. Explain how each element contributes to the reader's understanding.

Exercise 68:

A student has collected the following data for a cooling curve experiment and needs to present it in a report:

- 20 temperature measurements taken at 30-second intervals
- Each measurement has an uncertainty of ± 0.5°C
- The initial temperature was 95°C and the final was 25°C

Describe how this data should be presented in the Results section, including recommendations for tables, graphs, and uncertainty notation.

Exercise 69:

In an experiment to verify Snell's law, a student finds that the measured refractive index of glass is consistently 3% higher than the accepted value, despite the uncertainty being only 1%. Write a paragraph for the Discussion section that appropriately addresses this discrepancy, following the guidelines for speculation about model-system differences.

9. Appendix 1: The Gaussian Distribution - Mathematical Properties and Derivation

9.a. *The Equation of the Gaussian Distribution Curve*

Let's derive the equation that describes the Gaussian distribution, beginning with a fundamental model of random variation.

Consider a quantity whose true value is X, but when measured, it's subject to random uncertainty. We'll model this uncertainty as arising from many small, independent fluctuations that can be either positive or negative with equal probability.

Specifically, imagine that our measurement is affected by $2n$ small fluctuations, each with magnitude E. Each fluctuation has equal probability of being positive or negative. The measured value x can therefore range from $X - 2nE$ (if all fluctuations are negative) to $X + 2nE$ (if all are positive).

> **Note**
>
> Why this model makes sense
>
> This model reflects many real-world measurement situations. Think about measuring the length of an object with a ruler - your eye position, slight movements of your hand, tiny variations in lighting, and many other small factors all contribute small random errors to your measurement.

What we want to determine is the probability distribution for observing a particular deviation R within this range of possible values. This probability depends on how many different ways a specific deviation can occur.

9.a.i. *Understanding the Combinatorial Basis:*

Think about extreme deviations first. A deviation of exactly $+2nE$ can happen in only one way - when all $2n$ fluctuations are positive. Similarly, a deviation of $-2nE$ also happens in only one way.

A deviation of $(2n - 2)E$ is more likely because it can happen whenever exactly one of the fluctuations is negative (and the rest positive). Since any one of the $2n$ fluctuations could be that negative one, there are $2n$ different ways this deviation could occur.

More generally, if we want a total deviation R equal to $2rE$ (where $r \leq n$), this means that out of our $2n$ fluctuations, $(n + r)$ must be positive and $(n - r)$ must be negative. The number of ways to select $(n + r)$ positions from $2n$ positions is:

$$\frac{(2n)!}{(n+r)!(n-r)!} \tag{77}$$

This quantity represents the number of possible arrangements that yield our desired deviation. To convert this to a probability, we multiply by the probability of getting any specific arrangement of $(n + r)$ positive and $(n - r)$ negative fluctuations, which is:

$$\left(\frac{1}{2}\right)^{n+r}\left(\frac{1}{2}\right)^{n-r} = \left(\frac{1}{2}\right)^{2n} \tag{78}$$

The probability of deviation R is therefore:

$$\frac{(2n)!}{(n+r)!(n-r)!}\left(\frac{1}{2}\right)^{2n} \tag{79}$$

9.a.ii. *Simplifying with Stirling's Approximation:*

To evaluate our expression for large n, we need Stirling's approximation. Here's why this approximation works:

Consider that

$$\int_1^n \ln x \, dx = [x \ln x - x]_1^n = n \ln n - n + 1 \tag{80}$$

The integral approximates the sum $\ln 1 + \ln 2 + \ln 3 + \ldots + \ln n$, which equals $\ln(n!)$.

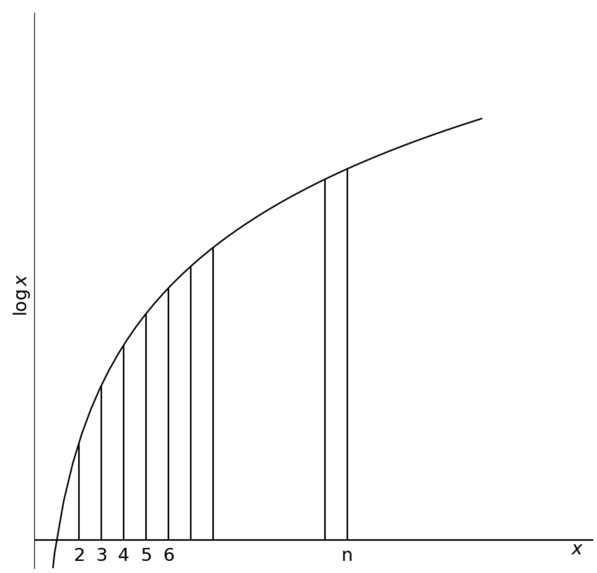

Figure 13: The area under the curve of ln(x) approximates the sum of logarithms, forming the basis for Stirling's approximation of n!.

Therefore:

$$\ln(n!) \approx n \ln n - n \tag{81}$$

$$n! \approx e^{-n} n^n \tag{82}$$

This gives us the basic form, though the complete approximation includes the $\sqrt{2\pi n}$ factor.

9.a.iii. *The Continuous Limit:*

Important

As n becomes very large, our discrete model approaches a continuous distribution - the Gaussian.

Applying Stirling's approximation to our probability expression and taking the limit as n approaches infinity (with appropriate simplifications that involve several algebraic steps), we eventually obtain:

$$\frac{1}{\sqrt{n\pi}} e^{-\frac{r^2}{n}} \tag{83}$$

This gives us the essence of the Gaussian form: **the probability decreases exponentially with the square of the deviation**. Converting to standard notation with x representing the deviation from the mean value X, and using a parameter h related to the width of the distribution:

$$P(x) = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2} \, dx \tag{84}$$

Where $P(x)dx$ represents the probability of finding a deviation between x and $x + dx$.

9.b. Standard Deviation of the Gaussian Distribution

The standard deviation provides a measure of the typical spread of values in the distribution. For a Gaussian distribution, we find the standard deviation by calculating:

$$\sigma^2 = \frac{1}{N} \int_{-\infty}^{\infty} \frac{Nh}{\sqrt{\pi}} e^{-h^2 x^2} x^2 \, dx = \frac{h}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 e^{-h^2 x^2} \, dx \tag{85}$$

Tip

Mathematical note

This integral can be evaluated using the formula:

$$\int_{-\infty}^{\infty} x^{2n} e^{-ax^2} \, dx = \frac{(2n-1)!!}{2^n a^n} \sqrt{\frac{\pi}{a}} \tag{86}$$

where $(2n-1)!! = (2n-1)(2n-3)...(3)(1)$

This integral equals $\frac{\sqrt{\pi}}{2h^3}$, giving us:

$$\sigma^2 = \frac{1}{2h^2} \tag{87}$$

Therefore:

$$\sigma = \frac{1}{\sqrt{2}h} \tag{88}$$

This allows us to rewrite the probability function in terms of the standard deviation:

$$P(x)dx = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \tag{89}$$

9.c. Areas Under the Gaussian Distribution Curve

A key practical question is: what fraction of measurements will fall within certain limits? To answer this, we need to find the area under portions of the Gaussian curve.

The probability that a measurement falls between 0 and x is:

$$\int_0^x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \tag{90}$$

Warning

This integral can't be evaluated in closed form (there's no elementary antiderivative). We must use numerical methods or look up values in tables.

This integral has been calculated numerically and tabulated. The table below shows these probabilities for different values of x/σ:

x/σ	Probability of deviation between 0 and x
0.0	0.0
0.5	0.19
1.0	0.34
1.5	0.43
2.0	0.48
3.0	0.499

Python Image: Click Me!

```
import numpy as np
import matplotlib.pyplot as plt
from scipy import stats
import matplotlib.patches as patches

# Create the figure and axis
plt.figure(figsize=(10, 6))
ax = plt.subplot(111)

# Define the x range and calculate the Gaussian PDF
```

```
x = np.linspace(-4, 4, 1000)
sigma = 1.0
mu = 0.0
pdf = 1/(sigma * np.sqrt(2*np.pi)) * np.exp(-(x-mu)**2/(2*sigma**2))

# Plot the Gaussian curve
plt.plot(x, pdf, 'k-', lw=2, label='Gaussian Distribution')

# Function to calculate the area (probability) between 0 and x_val
def area_between_0_and_x(x_val):
    return stats.norm.cdf(x_val) - stats.norm.cdf(0)

# Choose x/sigma value to illustrate - let's use x/sigma = 1.0
x_val = 1.0

# Fill the area from 0 to x_val
x_fill = np.linspace(0, x_val, 100)
y_fill = 1/(sigma * np.sqrt(2*np.pi)) * np.exp(-(x_fill-mu)**2/
(2*sigma**2))
plt.fill_between(x_fill, y_fill, color='skyblue', alpha=0.6)

# Add probability value text
prob = area_between_0_and_x(x_val)
plt.text(x_val/2, 0.15, f"Area = {prob:.3f}",
         ha='center', va='center', fontsize=12,
         bbox=dict(facecolor='white', alpha=0.8))

# Add x/sigma = 1.0 vertical line
plt.axvline(x=x_val, color='blue', linestyle='--', alpha=0.7)
plt.axvline(x=0, color='blue', linestyle='--', alpha=0.7)

# Annotate the endpoints
plt.annotate('x/σ = 0', xy=(0, 0), xytext=(0, -0.02),
             arrowprops=dict(arrowstyle='->'), ha='center')
plt.annotate(f'x/σ = {x_val}', xy=(x_val, 0), xytext=(x_val, -0.02),
             arrowprops=dict(arrowstyle='->'), ha='center')

# Add a small table showing values
table_data = [
    ['x/σ', 'Probability'],
    ['0.0', '0.000'],
    ['0.5', '0.192'],
    ['1.0', '0.341'],
    ['1.5', '0.433'],
    ['2.0', '0.477'],
    ['3.0', '0.499']
]

# Create the table in the upper right corner
table = plt.table(cellText=table_data, loc='upper right',
cellLoc='center',
```

```
                 colWidths=[0.1, 0.1], bbox=[0.7, 0.55, 0.28, 0.35])
table.auto_set_font_size(False)
table.set_fontsize(10)
table.scale(1, 1.5)

# Customize the plot
plt.grid(alpha=0.3)
plt.title('Area Under the Gaussian Distribution Curve', fontsize=14)
plt.xlabel('x/σ (Standard Deviations from Mean)', fontsize=12)
plt.ylabel('Probability Density', fontsize=12)
plt.xlim(-3, 3)
plt.ylim(-0.03, 0.45)

# Add a descriptive caption as text under the plot
plt.figtext(0.5, 0.01,
           "Figure A1.1: The shaded area represents the probability
of a \n"
           "deviation falling between 0 and x=1σ (34.1% of total
area).",
           ha='center', fontsize=11)

# Add an explanation of the concept
plt.text(-2.8, 0.3,
        "The probability that a measurement\n"
        "falls between 0 and x is given by:\n"
        r"$\int_0^x \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{t^2}
{2\sigma^2}}dt$",
        fontsize=10, bbox=dict(facecolor='lightyellow', alpha=0.9))

plt.tight_layout(rect=[0, 0.03, 1, 0.97])
plt.savefig('gaussian_area.svg', dpi=300)
plt.show()
```

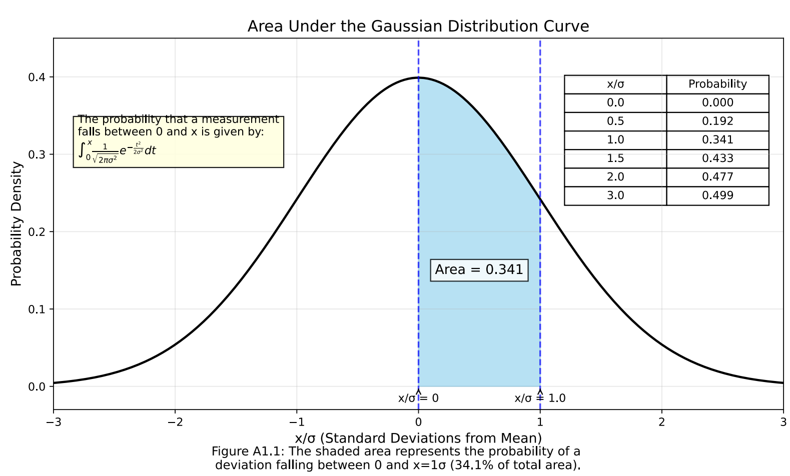

Figure 14: The shaded area under the Gaussian distribution curve represents the probability of a deviation falling between 0 and x. This integral cannot be evaluated in closed form and must be computed numerically.

For the probability that a measurement falls within $\pm x/\sigma$ of the mean (the symmetric interval), we double these values.

These probabilities form the foundation of statistical inference. When we make statements about the uncertainty of measurements, we often use these standard intervals - particularly the 68% confidence interval ($\pm 1\sigma$) and the 95% confidence interval ($\pm 2\sigma$).

10. Appendix 2: The Principle of Least Squares

10.a. Least Squares and Sample Means

Important

When we make multiple measurements of a quantity that contains random fluctuation, we need a method to determine the most probable value. The principle of least squares provides this method by finding the value that minimizes the squared deviations from our measurements.

Let's say we make N measurements, x_i, of a quantity. To find the value X whose deviations from our measurements are minimized according to the principle of least squares, we need:

$$\sum (x_i - X)^2 = \text{minimum} \tag{91}$$

Let's denote the mean of the measurements as $\bar{x}$. We can rewrite the sum of squared deviations as:

$$\sum (x_i - X)^2 = \sum [(x_i - \bar{x}) + (\bar{x} - X)]^2 \tag{92}$$

Expanding the squared term:

$$\sum (x_i - X)^2 = \sum \left[(x_i - \bar{x})^2 + (\bar{x} - X)^2 + 2(x_i - \bar{x})(\bar{x} - X)\right] \tag{93}$$

The cross-term $\sum(x_i - \bar{x})$ equals zero by definition of the mean, so:

$$\sum (x_i - X)^2 = \sum (x_i - \bar{x})^2 + N(\bar{x} - X)^2 \tag{94}$$

Note

This expression clearly reaches its minimum value when $X = \bar{x}$, confirming that using the sample mean as the most probable value is consistent with the principle of least squares.

10.b. Fitting a Straight Line Using Least Squares

Important

Consider a set of observations (x_i, y_i) that we wish to fit with a linear relationship:

$$y = mx + b \tag{95}$$

We'll assume that uncertainty exists only in the y values, and that all measurements have equal weight (we'll address weighted least squares later).

For each observation, the deviation from our proposed line is:

$$\delta y_i = y_i - (mx_i + b) \tag{96}$$

According to the principle of least squares, we want to minimize the sum of the squares of these deviations:

$$\sum (\delta y_i)^2 = \sum [y_i - (mx_i + b)]^2 \tag{97}$$

Expanding this expression:

$$\sum (\delta y_i)^2 = \sum [y_i^2 + m^2 x_i^2 + b^2 - 2mx_i y_i - 2by_i + 2mx_i b] \tag{98}$$

Or more compactly:

$$M = \sum y_i^2 + m^2 \sum x_i^2 + Nb^2 + 2mb \sum x_i - 2m \sum x_i y_i - 2b \sum y_i \tag{99}$$

Where M represents the sum of squared deviations that we want to minimize.

Tip

To find the optimal values of m and b, we take partial derivatives with respect to each parameter and set them equal to zero:

$$\frac{\partial M}{\partial m} = 0 \quad \text{and} \quad \frac{\partial M}{\partial b} = 0 \tag{100}$$

From the first condition:

$$2m \sum x_i^2 + 2b \sum x_i - 2 \sum (x_i y_i) = 0 \tag{101}$$

From the second condition:

$$2Nb + 2m \sum x_i - 2 \sum y_i = 0 \tag{102}$$

Solving these equations simultaneously gives us:

$$m = \frac{N \sum (x_i y_i) - \sum x_i \sum y_i}{N \sum x_i^2 - (\sum x_i)^2} \tag{103}$$

$$b = \frac{\sum x_i^2 \sum y_i - \sum x_i \sum (x_i y_i)}{N \sum x_i^2 - (\sum x_i)^2} \tag{104}$$

Important

Having determined the "best fit" line, we need to quantify the uncertainty in our calculated parameters. Since m and b are computed from measurements with uncertainty, we can calculate their standard deviations.

For the standard deviation of each y_i value from our fitted line, we use:

$$S_y = \sqrt{\frac{\sum (\delta y_i)^2}{N - 2}} \tag{105}$$

Note

The denominator uses $N - 2$ rather than N because we've estimated two parameters (m and b) from our data, reducing our degrees of freedom.

The standard deviations of the slope and intercept are then:

$$S_m = S_y \sqrt{\frac{N}{N \sum x_i^2 - \left(\sum x_i\right)^2}} \tag{106}$$

$$S_b = S_y \sqrt{\frac{\sum x_i^2}{N \sum x_i^2 - \left(\sum x_i\right)^2}} \tag{107}$$

Tip

These expressions provide statistical measures of uncertainty in our fitted parameters. When reporting results, we typically state values as $m \pm S_m$ and $b \pm S_b$, indicating that the true parameter has about a 68% probability of falling within one standard deviation of our estimate.

10.c. *Weighted Least Squares*

Important

When measurements have different levels of precision, it makes sense to give more weight to more precise measurements. This approach is called weighted least squares.

10.c.i. *Weighted Mean of Observations:*

Note

If we have independently measured quantities x_i, each with a standard deviation S_i, the weighted mean is:

$$\bar{x} = \frac{\sum (x_i / S_i^2)}{\sum (1 / S_i^2)} \tag{108}$$

The standard deviation of this weighted mean is:

$$S^2 = \frac{\sum \left((x_i - \bar{x})^2 / S_i^2 \right)}{(N-1) \sum (1 / S_i^2)} \tag{109}$$

10.c.ii. *Straight-Line Fitting with Weighted Least Squares:*

Important

For observations with unequal precision, we modify our least squares approach by assigning weights. If the y values have varying precision, but the x values are considered exact, the equations for the slope and intercept become:

$$m = \frac{\sum w_i \sum w_i x_i y_i - \sum w_i x_i \sum w_i y_i}{\sum w_i \sum w_i x_i^2 - \left(\sum w_i x_i\right)^2} \tag{110}$$

$$b = \frac{\sum w_i y_i \sum w_i x_i^2 - \sum w_i x_i \sum w_i x_i y_i}{\sum w_i \sum w_i x_i^2 - \left(\sum w_i x_i\right)^2} \quad (111)$$

Where w_i represents the weight of each observation, calculated as:

$$w_i = \frac{1}{S_{yi}^2} \quad (112)$$

The weighted standard deviation about the best-fit line is:

$$S_y = \sqrt{\frac{\sum w_i \delta_i^2}{N-2}} \quad (113)$$

And the standard deviations of the slope and intercept are:

$$S_m^2 = \frac{S_y^2}{W} \quad (114)$$

$$S_b^2 = S_y^2 \left(\frac{1}{\sum w_i} + \frac{\bar{x}^2}{W} \right) \quad (115)$$

Where:

$$W = \sum \left(w_i (x_i - \bar{x})^2 \right) \quad (116)$$

And $\bar{x}$ is the weighted mean of the x values:

$$\bar{x} = \frac{\sum w_i x_i}{\sum w_i} \quad (117)$$

Tip

Weighted least squares is particularly valuable when measurements come from different sources with varying precision. By accounting for these differences in precision, we ensure that our fitted parameters are not unduly influenced by less reliable data points.

Warning

When reporting results from weighted analyses, it's important to specify that weighted methods were used and to explain the basis for the weights assigned. This transparency allows others to properly interpret and potentially reproduce your analysis.

11. Appendix 3: Introduction to Jupyter and Python for Data Analysis

11.a. *The Power of Computational Tools in Modern Physics*

Modern physics laboratories have been transformed by computational tools that allow for efficient data processing, powerful visualizations, and sophisticated statistical analysis. Two of the most valuable tools in a physicist's computational toolkit are Python and Jupyter Notebooks.

> **Note**
>
> This appendix assumes no prior programming experience. We'll start with the fundamentals and build toward practical applications for physics laboratory data analysis.

11.b. *Getting Started with Python*

Python has become the de facto standard programming language for scientific computing due to its readability, extensive scientific libraries, and supportive community.

11.b.i. *Installing Python:*

Before diving into coding, you'll need to set up a Python environment:

> **Tip**
>
> The easiest way to get started is by installing Anaconda, a distribution that includes Python, Jupyter, and many scientific packages. Download it from anaconda.com.

Once installed, you can verify your installation by opening a terminal (Command Prompt on Windows, Terminal on macOS/Linux) and typing:

```
python --version
```

11.b.ii. *Python Fundamentals:*

Python is a high-level, interpreted programming language known for its clear syntax. Let's explore some basics:

Variables and Data Types:

```
# Numbers
mass = 9.8  # A floating-point number (decimal)
count = 5   # An integer

# Strings (text)
element = "Hydrogen"

# Boolean values
is_valid = True
```

```
# Lists (ordered collections)
readings = [9.81, 9.79, 9.82, 9.80]

# Printing values
print(f"The mass is {mass} kg")
print(f"We took {count} measurements")
print(f"The mean of our readings is {sum(readings)/len(readings)}")
```

Basic Math Operations:

Python handles mathematical operations naturally:

```
# Basic arithmetic
a = 10
b = 3

print(a + b)  # Addition: 13
print(a - b)  # Subtraction: 7
print(a * b)  # Multiplication: 30
print(a / b)  # Division: 3.3333...
print(a ** b) # Exponentiation: 1000
print(a % b)  # Modulo (remainder): 1

# Using the math library for more advanced functions
import math

angle = math.pi/4  # 45 degrees in radians
print(math.sin(angle))  # Sine function
print(math.sqrt(16))    # Square root: 4.0
```

Control Flow:

Python uses indentation to define code blocks for conditions and loops:

```
# Conditional statements
temperature = 22.5

if temperature > 25:
    print("It's warm")
elif temperature < 15:
    print("It's cold")
else:
    print("It's a pleasant temperature")

# Loops
print("Measuring temperatures:")
temperatures = [22.1, 22.4, 22.3, 22.5, 22.2]
for temp in temperatures:
    print(f"Reading: {temp}°C")

# While loops
count = 0
while count < 5:
```

```
    print(f"Count: {count}")
    count += 1  # Shorthand for count = count + 1
```

Warning

Indentation in Python is not just for readability—it's how Python identifies code blocks. Inconsistent indentation will cause syntax errors.

Functions:

Functions allow you to encapsulate reusable code blocks:

```
def calculate_kinetic_energy(mass, velocity):
    """Calculate kinetic energy using the formula E = 1/2 * m * v^2."""
    return 0.5 * mass * velocity**2

# Using the function
mass = 0.5  # kg
velocity = 10  # m/s
energy = calculate_kinetic_energy(mass, velocity)
print(f"The kinetic energy is {energy} J")
```

Note

The triple-quoted string after the function definition is called a "docstring." It documents the function's purpose and is good practice in scientific code.

11.c. *Scientific Computing Libraries*

Python's real power for physics comes from its scientific computing ecosystem:

11.c.i. *NumPy: Numerical Python:*

NumPy provides support for arrays, matrices, and many mathematical functions:

```
import numpy as np

# Creating arrays
data = np.array([1.2, 2.3, 3.4, 4.5, 5.6])
print(f"Mean: {np.mean(data)}")
print(f"Standard deviation: {np.std(data)}")

# Array operations (vectorized calculations)
scaled_data = 2 * data  # Multiplies each element by 2
shifted_data = data + 10  # Adds 10 to each element

# Creating a range of values (useful for x-axes)
time = np.linspace(0, 10, 100)  # 100 points from 0 to 10
position = 4.9 * time**2  # Position in free fall
```

11.c.ii. *Matplotlib: Visualization:*

Matplotlib creates publication-quality graphs:

```
import matplotlib.pyplot as plt

# Simple plot
plt.figure(figsize=(8, 6))  # Set figure size in inches
plt.plot(time, position, 'b-', label='Position')  # 'b-' means blue line
plt.xlabel('Time (s)')
plt.ylabel('Position (m)')
plt.title('Free Fall Motion')
plt.grid(True)
plt.legend()
plt.show()

# Scatter plot with error bars
x = np.array([1, 2, 3, 4, 5])
y = np.array([2.1, 3.9, 6.2, 7.8, 10.1])
y_error = np.array([0.2, 0.3, 0.2, 0.4, 0.3])

plt.figure(figsize=(8, 6))
plt.errorbar(x, y, yerr=y_error, fmt='ro', capsize=5,
label='Measurements')
plt.xlabel('Input Variable')
plt.ylabel('Output Variable')
plt.title('Experiment Results with Error Bars')
plt.grid(True)
plt.legend()
plt.show()
```

11.c.iii. *SciPy: Scientific Python:*

SciPy extends NumPy with additional scientific functionality:

```
from scipy import stats
from scipy.optimize import curve_fit

# Linear regression
slope, intercept, r_value, p_value, std_err = stats.linregress(x, y)
print(f"Slope: {slope:.4f} ± {std_err:.4f}")
print(f"Intercept: {intercept:.4f}")
print(f"R-squared: {r_value**2:.4f}")

# Curve fitting
def model_func(x, a, b):
    """Model y = a*x^b."""
    return a * x**b

params, params_covariance = curve_fit(model_func, x, y)
a, b = params
print(f"Fitted parameters: a = {a:.4f}, b = {b:.4f}")
```

11.c.iv. *Pandas: Data Manipulation:*

Pandas excels at handling structured data like CSV files:

```python
import pandas as pd

# Reading data from CSV file
df = pd.read_csv('experiment_data.csv')
print(df.head())  # Print first few rows

# Basic statistics
print(df.describe())

# Selecting columns
time_data = df['Time']
position_data = df['Position']

# Filtering data
filtered_data = df[df['Temperature'] > 25]
```

11.d. Introduction to Jupyter Notebooks

Jupyter Notebooks provide an interactive environment that combines code, text, equations, and visualizations.

11.d.i. *Starting Jupyter:*

To start Jupyter Notebook, open your terminal and run:

```bash
jupyter notebook
```

This will open a web browser showing the Jupyter dashboard, where you can create or open notebooks.

11.d.ii. *Notebook Components:*

A Jupyter Notebook consists of cells that can contain:

1. **Code** - Python code that can be executed
2. **Markdown** - Text with formatting, equations, and links
3. **Raw** - Plain text without formatting

> **Tip**
>
> Press `Shift+Enter` to run a cell and move to the next one. Press `Ctrl+Enter` to run a cell and stay on it.

11.d.iii. *Markdown and LaTeX in Jupyter:*

Jupyter supports Markdown for text formatting and LaTeX for equations:

```markdown
# Heading 1

## Heading 2

_italic text_
**bold text**
```

```
- Bullet point
- Another point

1. Numbered item
2. Another item

[Link text](https://example.com)

Inline equation: $E = mc^2$

Display equation:
$$F = G\frac{m_1 m_2}{r^2}$$
```

> **Important**
>
> The ability to mix explanatory text, mathematical equations, code, and visualizations makes Jupyter ideal for documenting laboratory experiments and analysis.

11.e. Practical Example: Analyzing Pendulum Data

Let's walk through a complete example of analyzing pendulum period data using Python and Jupyter.

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import curve_fit
import pandas as pd

# Sample data: pendulum length (m) and period (s)
length = np.array([0.10, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40, 0.45, 0.50])
period = np.array([0.64, 0.78, 0.90, 1.01, 1.11, 1.19, 1.28, 1.36, 1.43])
period_error = np.array([0.02, 0.02, 0.02, 0.03, 0.03, 0.03, 0.03, 0.04, 0.04])

# Create a DataFrame for better data organization
data = pd.DataFrame({
    'Length (m)': length,
    'Period (s)': period,
    'Period Error (s)': period_error
})

print("Pendulum Data:")
print(data)

# Calculate the square of the period
data['Period² (s²)'] = data['Period (s)']**2
data['Period² Error (s²)'] = 2 * data['Period (s)'] * data['Period Error (s)']

# Define the theoretical model: T² = (4π²/g) * L
```

```python
def model_function(L, g):
    """Model the relationship T² = (4π²/g) * L."""
    return (4 * np.pi**2 / g) * L

# Perform the curve fitting
params, params_covariance = curve_fit(
    model_function,
    data['Length (m)'],
    data['Period² (s²)'],
    sigma=data['Period² Error (s²)'],
    absolute_sigma=True
)

# Extract the fitted parameter (g) and its uncertainty
g_fitted = params[0]
g_error = np.sqrt(params_covariance[0, 0])

print(f"\nFitted value of g: {g_fitted:.3f} ± {g_error:.3f} m/s²")

# Create a nice plot
plt.figure(figsize=(10, 6))

# Plot the data points with error bars
plt.errorbar(
    data['Length (m)'],
    data['Period² (s²)'],
    yerr=data['Period² Error (s²)'],
    fmt='o',
    markersize=6,
    capsize=3,
    label='Experimental data'
)

# Plot the best fit line
L_values = np.linspace(0, 0.55, 100)
T2_fitted = model_function(L_values, g_fitted)
plt.plot(L_values, T2_fitted, 'r-',
         label=f'Best fit: T² = (4π²/g) * L, g = {g_fitted:.3f} m/s²')

# Add the expected line for g = 9.81 m/s²
T2_expected = model_function(L_values, 9.81)
plt.plot(L_values, T2_expected, 'g--',
         label=f'Expected: g = 9.81 m/s²')

# Customize the plot
plt.xlabel('Pendulum Length (m)')
plt.ylabel('Period² (s²)')
plt.title('Pendulum Period² vs. Length')
plt.grid(True, alpha=0.3)
plt.legend()
plt.tight_layout()
plt.show()
```

```
# Calculate the residuals (difference between observed and fitted
values)
data['Fitted Period² (s²)'] = model_function(data['Length (m)'],
g_fitted)
data['Residual (s²)'] = data['Period² (s²)'] - data['Fitted Period²
(s²)']

# Plot the residuals
plt.figure(figsize=(10, 4))
plt.errorbar(
    data['Length (m)'],
    data['Residual (s²)'],
    yerr=data['Period² Error (s²)'],
    fmt='o',
    markersize=6,
    capsize=3
)
plt.axhline(y=0, color='r', linestyle='-')
plt.xlabel('Pendulum Length (m)')
plt.ylabel('Residual (s²)')
plt.title('Residuals of the Fit')
plt.grid(True, alpha=0.3)
plt.tight_layout()
plt.show()
```

Note

This example demonstrates how to:

- Organize experimental data
- Propagate uncertainties
- Fit a theoretical model to data
- Visualize results with appropriate error bars
- Compare fitted results with expected values
- Analyze residuals to evaluate the quality of the fit

11.f. Data Analysis Workflow for Physics Experiments

When approaching data analysis for physics experiments, a systematic workflow is helpful:

1. **Data Import and Organization**
    - Import raw data (CSV, Excel, etc.)
    - Organize into appropriate data structures
    - Perform basic validation and cleaning
2. **Exploratory Analysis**
    - Calculate basic statistics (mean, standard deviation)
    - Create initial visualizations
    - Identify potential outliers or issues

3. **Data Transformation**
    - Convert units if necessary
    - Create derived quantities
    - Apply appropriate transformations (e.g., linearization)
4. **Model Fitting**
    - Define theoretical model
    - Perform regression or curve fitting
    - Extract parameters and their uncertainties
5. **Visualization**
    - Create publication-quality plots
    - Include error bars and uncertainty ranges
    - Compare experimental results with theoretical predictions
6. **Result Analysis and Interpretation**
    - Evaluate goodness of fit
    - Analyze residuals
    - Calculate derived quantities with proper uncertainty propagation
    - Compare results with established values or theories

> **Tip**
>
> For reproducibility, document each step of your analysis in your Jupyter Notebook with clear markdown explanations. This makes it easier to trace your reasoning and catch potential errors.

11.g. *Tips for Efficient Data Analysis*

11.g.i. *Coding Best Practices:*

1. **Comment your code**

```
# Calculate gravitational acceleration from the slope
g = 4 * np.pi**2 / slope  # Converting from T² vs L to g
```

1. **Use descriptive variable names**

```
# Good
time_of_flight = distance_fallen / initial_velocity

# Avoid
t = d / v
```

1. **Structure your notebook logically**
    - Start with imports and setup
    - Follow with data loading and processing
    - Continue with analysis and visualization
    - End with conclusions

11.g.ii. *Data Visualization Tips:*

1. **Always label your axes**

```
plt.xlabel('Time (s)')
plt.ylabel('Displacement (m)')
```

1. **Include units in labels**

```
plt.xlabel('Pressure (kPa)')
```

1. **Use appropriate scales**

```
plt.xscale('log')  # For logarithmic scale
```

1. **Add error bars when possible**

```
plt.errorbar(x, y, yerr=y_errors, fmt='o')
```

1. **Include a legend when plotting multiple series**

```
plt.plot(x, y1, 'b-', label='Measured')
plt.plot(x, y2, 'r--', label='Theoretical')
plt.legend()
```

11.g.iii. *Handling Experimental Uncertainties:*

1. **Propagate uncertainties correctly**

```
# For y = a*x + b
y_error = np.sqrt((a_error * x)**2 + b_error**2)
```

1. **Use weighted fits when measurement uncertainties vary**

```
weights = 1 / (y_errors**2)
params, params_covariance = curve_fit(model, x, y, sigma=y_errors,
absolute_sigma=True)
```

1. **Check residuals for patterns**

```
residuals = y_data - model(x_data, *params)
plt.plot(x_data, residuals, 'o')
```

11.h. Advanced Topics

11.h.i. *Automating Repetitive Tasks:*

When processing multiple datasets with similar structure, functions can help automate the work:

```
def analyze_pendulum_data(filepath, output_folder=None):
    """Analyze pendulum data from a CSV file.

    Parameters:
    -----------
    filepath : str
        Path to the CSV file containing length and period data
    output_folder : str, optional
        Folder to save output plots

    Returns:
```

```
    --------
    dict
        Dictionary containing analysis results
    """
    # Load data
    data = pd.read_csv(filepath)

    # Perform analysis
    # ...

    # Create and save plots
    # ...

    return results
```

11.h.ii. *Saving and Sharing Your Work:*

Jupyter Notebooks can be shared in various ways:

1. **Export as HTML, PDF, or other formats**
   - In Jupyter: File > Export Notebook As...
2. **Version control with Git/GitHub**
   - Notebooks are text files that can be tracked with version control
   - GitHub renders notebooks directly in the browser
3. **Interactive sharing with Binder**
   - Share executable versions of your notebooks online

> **Important**
>
> When sharing your analysis, include the raw data files or clear instructions on how to obtain them.

11.i. Glossary

11.j. Problems

Exercise 70:

A student measures the terminal velocity of different objects falling through a viscous fluid. The data is stored in a CSV file with columns for 'Radius (mm)', 'Mass (g)', and 'Velocity (cm/s)'. Write Python code to:

a. Load the data b. Convert to SI units c. Test if the velocity is proportional to the square of the radius, as predicted by Stokes' Law d. Determine the viscosity of the fluid

Exercise 71:

Using the pendulum example from this appendix, modify the code to:

a. Add a random error to each period measurement b. Run the analysis 1000 times with different random errors c. Create a histogram of the resulting g values d. Determine if the uncertainty estimated by curve_fit matches the standard deviation of your Monte Carlo simulation

Exercise 72:

A student measures the distance vs. time for a cart rolling down an inclined plane. Write code to:

a. Create a scatter plot of distance vs. time b. Fit both a linear model ($d = vt$) and a quadratic model ($d = \frac{1}{2}at^2$) c. Compare the models using residual analysis d. Determine which model better describes the motion

Exercise 73:

Create a Jupyter notebook that demonstrates the propagation of uncertainties for different mathematical operations (addition, multiplication, powers, etc.) using both analytical formulas and Monte Carlo simulation.

Exercise 74:

Using the least squares method described in Appendix 2, implement the weighted least squares algorithm in Python and compare its results with those from SciPy's curve_fit function.

12. Appendix 4: Model Experiment

12.-.i. *Experiment Design:*

System:

For our experiment, we have assembled the following apparatus:

- A helical spring suspended from a rigid laboratory stand with vibration-dampening clamps
- A precision-machined pan for holding weights, attached to the lower end of the spring via a low-friction hook
- A set of calibrated brass weights (class M1 standard, $\pm$0.1mg tolerance)
- A digital stopwatch with millisecond precision (systematic uncertainty $\pm$0.01s)
- A meter rule with millimeter graduations for measuring displacements
- A digital camera capable of high-speed recording (120fps) for motion analysis verification

Model:

According to fundamental principles of classical mechanics, a spring's extension is proportional to the applied load when operating within its elastic limit (Hooke's Law). For a mass-spring system undergoing simple harmonic motion, the period of oscillation (T) relates to the suspended mass (m) through the equation:

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{118}$$

Where k represents the spring constant measured in N/m (or equivalently, kg/s²).

Requirement:

Our experimental objective is to determine the spring constant k with an uncertainty not exceeding 10%. This precision requirement guides our experimental design and measurement protocols.

Experiment Design:

We employ a systematic approach to experimental design following established best practices:

1. **System identification and isolation**: We carefully isolate the spring-mass system, minimizing external influences such as air currents and vibrations by using a vibration-dampening table and conducting measurements in a temperature-controlled environment.
2. **Variable selection and control**: We identify two key measurable variables—the load m (independent variable we systematically vary) and the period of oscillation T (dependent variable we measure).
3. **Mathematical model transformation**: To facilitate statistical analysis, we transform our physical model into a linear relationship:

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{119}$$

Squaring both sides:

$$T^2 = \frac{4\pi^2}{k}m \tag{120}$$

This can be represented in slope-intercept form:

$$m = \frac{k}{4\pi^2}T^2 + b \tag{121}$$

Where:

- Vertical axis variable = m
- Horizontal axis variable = T^2
- Slope = $\frac{k}{4\pi^2}$
- Intercept = b (theoretically zero, but may reveal systematic effects)

This transformation allows us to determine k directly from the slope using linear regression techniques.

1. **Measurement range optimization**: We carefully consider:
    - The available calibrated weights (0.05 kg to 0.50 kg)
    - The spring's elastic limit (determined through preliminary testing to be approximately 0.60 kg)
    - Practical constraints on timing oscillations (targeting relative timing uncertainty <1%)
    - Signal-to-noise ratio optimization (larger masses produce longer periods, reducing relative timing uncertainty)
2. **Uncertainty propagation analysis**: For a 10% maximum uncertainty in k, we conduct uncertainty propagation analysis:

For time measurements with digital stopwatch uncertainty of ± 0.01s, we need to minimize the relative uncertainty in period measurements. Since the uncertainty in T^2 is approximately twice the relative uncertainty in T, we require:

$$\frac{\Delta T}{T} < 0.05 \tag{122}$$

For a conservative uncertainty estimate of ± 0.02s per period measurement:

$$\frac{0.02 \text{ s}}{T} < 0.05 \tag{123}$$

Which yields:

$$T > 0.4 \quad \text{seconds} \tag{124}$$

To further reduce uncertainty, we time multiple oscillations (n=10) and calculate:

$$T = \frac{t_{total}}{n} \tag{125}$$

This reduces timing uncertainty by a factor of approximately √n.

Measurement Protocol:

We developed a comprehensive measurement protocol:

1. **System calibration**:
   - Zero the digital scale used to verify weights
   - Calibrate the digital stopwatch against a reference timekeeper
   - Measure the unloaded spring length as reference
2. **Data collection procedure**:
   - Attach the weight pan (mass recorded separately)
   - Add calibrated weights incrementally
   - For each load, displace the system 2 cm from equilibrium
   - Release from rest and time 10 complete oscillations
   - Repeat measurements three times per load setting
   - Record ambient temperature and pressure
3. **Data processing methodology**:
   - Calculate average period and associated uncertainty for each load
   - Compute T^2 values and propagate uncertainties
   - Plot m versus T^2 with error bars
   - Perform weighted least-squares regression analysis using Python

This structured methodology ensures reproducibility and minimizes both random and systematic uncertainties in our measurements.

12.-.ii. *Experimental Results:*

Raw Measurements:

The measurements are presented in Table 1, with each entry including its associated uncertainty determined through statistical analysis of repeated measurements.

Load, m (kg)	# of Osc.	Time, t (s)	Period, T (s)	Period², T^2 (s^2)	ΔT^2 (s^2)
0.10 ± 0.0001	10	8.20 ± 0.03	0.820 ± 0.003	0.672	0.005
0.15 ± 0.0001	10	9.80 ± 0.03	0.980 ± 0.003	0.960	0.006
0.20 ± 0.0001	10	10.70 ± 0.03	1.070 ± 0.003	1.145	0.006
0.25 ± 0.0001	10	11.50 ± 0.03	1.150 ± 0.003	1.323	0.007
0.30 ± 0.0001	10	12.50 ± 0.03	1.250 ± 0.003	1.563	0.008
0.35 ± 0.0001	10	13.00 ± 0.03	1.300 ± 0.003	1.690	0.008
0.40 ± 0.0001	10	13.80 ± 0.03	1.380 ± 0.003	1.904	0.008
0.45 ± 0.0001	10	14.50 ± 0.03	1.450 ± 0.003	2.103	0.009
0.50 ± 0.0001	10	15.20 ± 0.03	1.520 ± 0.003	2.310	0.009

Table 2: Variation of Oscillation Period with Load

Computational Analysis:

We performed data analysis using Python with NumPy and SciPy libraries. Below is the analysis script used to process our experimental data:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import curve_fit
from scipy import stats

# Load experimental data
masses = np.array([0.10, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40, 0.45,
0.50])
periods_squared = np.array([0.672, 0.960, 1.145, 1.323, 1.563, 1.690,
1.904, 2.103, 2.310])
uncertainties = np.array([0.005, 0.006, 0.006, 0.007, 0.008, 0.008,
0.008, 0.009, 0.009])

# Define linear model function
def linear_model(x, slope, intercept):
    return slope * x + intercept

# Perform weighted least-squares fit
weights = 1 / (uncertainties**2)
popt, pcov = curve_fit(linear_model, periods_squared, masses,
                        sigma=uncertainties, absolute_sigma=True)

slope, intercept = popt
slope_err, intercept_err = np.sqrt(np.diag(pcov))

# Calculate spring constant and its uncertainty
k = 4 * np.pi**2 * slope
k_err = 4 * np.pi**2 * slope_err

# Calculate coefficient of determination (R²)
residuals = masses - linear_model(periods_squared, *popt)
ss_res = np.sum(residuals**2)
ss_tot = np.sum((masses - np.mean(masses))**2)
r_squared = 1 - (ss_res / ss_tot)

# Generate prediction intervals (95% confidence)
t_value = stats.t.ppf(0.975, len(masses)-2)
prediction_intervals = t_value * np.sqrt(1/weights +
                      (periods_squared - np.mean(periods_squared))**2 /
                      np.sum(weights * (periods_squared -
np.mean(periods_squared))**2))

# Plot results with error bars and confidence intervals
plt.figure(figsize=(10, 7))
plt.errorbar(periods_squared, masses, xerr=uncertainties, fmt='o',
```

```
             markersize=6, capsize=3, label='Experimental data')

# Plot best fit line
x_fit = np.linspace(0.5, 2.5, 100)
plt.plot(x_fit, linear_model(x_fit, *popt), 'r-',
         label=f'Best fit: m = ({slope:.4f}±{slope_err:.4f})T² +
({intercept:.4f}±{intercept_err:.4f})')

# Plot prediction intervals
plt.fill_between(periods_squared,
                 linear_model(periods_squared, *popt) -
prediction_intervals,
                 linear_model(periods_squared, *popt) +
prediction_intervals,
                 alpha=0.2, color='gray', label='95% confidence
interval')

plt.xlabel('Period squared, T² (s²)')
plt.ylabel('Mass, m (kg)')
plt.title('Determination of Spring Constant via Oscillation Method')
plt.grid(True, alpha=0.3)
plt.legend()
plt.savefig('spring_constant_analysis.png', dpi=300)
plt.show()

print(f"Spring constant k = {k:.2f} ± {k_err:.2f} N/m")
print(f"Coefficient of determination R² = {r_squared:.6f}")
print(f"Y-intercept = {intercept:.4f} ± {intercept_err:.4f} kg")
```

The analysis yielded a coefficient of determination (R^2) of 0.9996, indicating an excellent fit to our linear model.

Results Visualization:

Figure 1 shows the results of our computational analysis, including the experimental data points with uncertainties, the best-fit line, and the 95% confidence intervals derived from our statistical analysis.

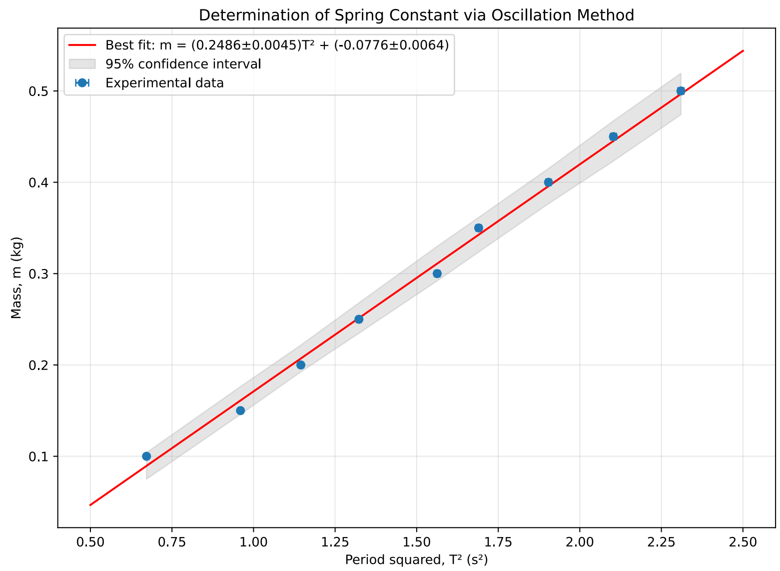

Figure 15: Plot of mass versus period squared showing experimental data points with uncertainties, best-fit line determined by weighted least-squares regression, and 95% confidence intervals.

Parameter Determination:

From our computational analysis, we obtained:

Best-fit parameters:

- Slope = 0.2486 ± 0.0045 kg/s²
- Intercept = -0.0776 ± 0.0064 kg

Derived spring constant:

$$k = 4\pi^2 \times \text{slope} = 4\pi^2 \times 0.2486 = 9.81 \ \text{ N }\backslash/\text{ m} \tag{126}$$

Uncertainty propagation:

$$\Delta k = 4\pi^2 \times \Delta \text{ slope} = 4\pi^2 \times 0.0045 = 0.18 \ \text{ N }\backslash/\text{ m} \tag{127}$$

Our final result is:

$$k = 9.81 \pm 0.18 \ \text{ N }\backslash/\text{ m} \tag{128}$$

This gives us a relative uncertainty of 1.8%, significantly better than our target of 10%.

12.-.iii. *Extended Procedure:*

Our experimental procedure followed these detailed steps:

1. **Equipment preparation and verification:**

  - The spring was examined for damage or permanent deformation
  - The spring was pre-stretched with a 0.6 kg load for 30 minutes to minimize hysteresis effects
  - Weight calibration was verified using an analytical balance (±0.1 mg precision)
  - The support stand was secured to a vibration-isolated optical table
  - Level adjustment was performed using a spirit level
2. **Environmental control:**
  - Room temperature maintained at 22.0 ± 0.5°C
  - Airflow minimized by closing vents and doors
  - Barometric pressure recorded (101.3 kPa)
  - Relative humidity monitored (45%)
3. **Preliminary measurements:**
  - The unloaded length of the spring was measured (15.3 ± 0.1 cm)
  - The mass of the empty pan was determined (0.023 ± 0.0001 kg)
  - The spring's elastic limit was assessed through static loading tests
  - Natural frequency of the laboratory bench was measured to identify potential resonance issues
4. **Measurement procedure:**
  - The pan was attached to the spring and allowed to reach equilibrium
  - The initial position was marked on a background grid for reference
  - Calibrated weights were added incrementally (0.05 kg steps)
  - For each load configuration:
    - The system was displaced 2.0 cm downward using a release mechanism
    - A digital stopwatch was used to time 10 complete oscillations
    - The measurement was repeated three times with brief pauses between trials
    - The system was allowed to return to equilibrium before the next trial
    - Any observed damping was noted qualitatively
  - High-speed video (120 fps) recorded select trials for verification
  - Between measurement sets, the spring was inspected for signs of fatigue
5. **Data analysis methodology:**
  - Statistical treatment applied to repeated measurements:
    - Mean values calculated for each measurement set
    - Standard deviation determined as a measure of random uncertainty
    - Standard error of the mean computed for each average period
  - Systematic uncertainties identified and quantified:
    - Stopwatch calibration uncertainty (±0.01s)
    - Mass calibration uncertainty (±0.0001 kg)

    - Human reaction time variation (minimized through training)
  - Computational analysis performed using Python libraries:
    - NumPy for numerical operations
    - SciPy.optimize for curve fitting with weighted least-squares
    - Matplotlib for visualization with error representation
  - Uncertainty propagation calculated following standard error propagation formulas
  - Goodness-of-fit evaluated using coefficient of determination (R^2)
  - Residual analysis performed to check for systematic patterns
6. **Verification methods:**
  - Selected trials analyzed frame-by-frame using video analysis software
  - Static loading tests performed to cross-verify spring constant
  - Amplitude independence verified by varying initial displacement
  - Zero-crossing method used as alternative timing approach for validation

This comprehensive procedure ensured high-quality data collection with minimized uncertainties and thorough validation of our results.

12.-.iv. *Report:*

12.a. MEASUREMENT OF A SPRING CONSTANT BY AN OSCILLATION METHOD

Introduction:

The stiffness of a spring, characterized by its spring constant (k), represents a fundamental physical parameter with applications ranging from engineering design to theoretical mechanics. For an elastic spring operating within Hooke's Law, the period of oscillation (T) of a suspended mass (m) follows the relationship:

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{129}$$

This experiment employs modern computational methods to determine the spring constant with high precision, aiming for an uncertainty below 10%. By transforming the equation into a linear form:

$$m = \frac{k}{4\pi^2}T^2 + b \tag{130}$$

We can apply weighted least-squares regression analysis to determine k from the slope of the m vs. T^2 relationship, while also investigating potential systematic effects revealed by any non-zero intercept.

Procedure:

We established a precision measurement system consisting of a helical spring suspended from a vibration-isolated support structure (Figure 2). The experimental apparatus included:

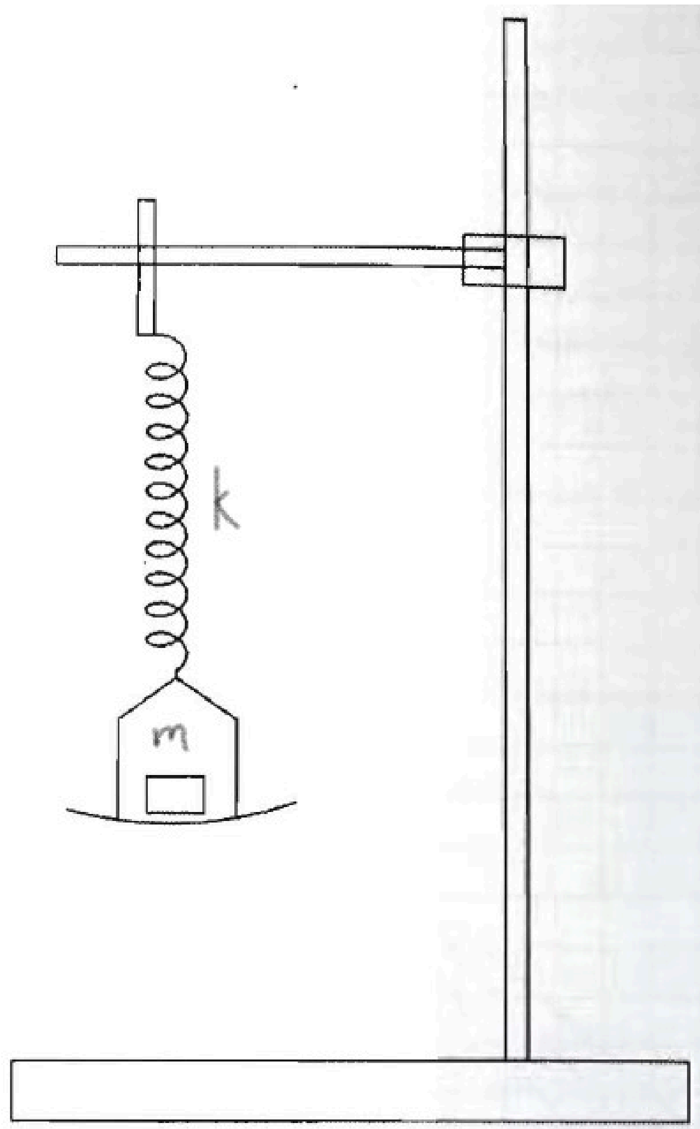

Figure 16: Schematic diagram of the experimental apparatus showing the spring suspension system, digital measurement tools, and vibration isolation measures.

The apparatus featured:

- A class-2 helical spring (wire diameter 0.8mm, mean coil diameter 10mm, 35 active coils)
- Calibrated M1-class brass weights (0.05kg to 0.50kg, $\pm$0.1mg tolerance)
- Lightweight aluminum pan (23.0g) with three-point suspension
- Digital stopwatch with millisecond resolution
- Meter rule with vernier scale for displacement measurements
- High-speed camera (120fps) for motion verification
- Temperature and humidity monitoring systems
- Vibration-isolated optical table

Our measurement protocol involved:

1. Suspending the spring from the support stand and attaching the weight pan
2. Adding calibrated weights incrementally from 0.10kg to 0.50kg
3. Displacing the system 2.0cm from equilibrium using a release mechanism
4. Timing ten complete oscillations for each load configuration
5. Repeating measurements three times per configuration to assess repeatability
6. Recording environmental conditions throughout the experiment

Data analysis employed numerical methods using Python with scientific computing libraries, applying weighted least-squares regression to determine the spring constant and its associated uncertainty.

Results:

The measured relationship between load and oscillation period is presented in Table 2, with uncertainties determined through statistical analysis of repeated measurements.

Load, m (kg)	# of Osc.	Time, t (s)	Period, T (s)	Period², T² (s²)
0.10 ± 0.0001	10	8.20 ± 0.03	0.820 ± 0.003	0.672 ± 0.005
0.15 ± 0.0001	10	9.80 ± 0.03	0.980 ± 0.003	0.960 ± 0.006
0.20 ± 0.0001	10	10.70 ± 0.03	1.070 ± 0.003	1.145 ± 0.006
0.25 ± 0.0001	10	11.50 ± 0.03	1.150 ± 0.003	1.323 ± 0.007
0.30 ± 0.0001	10	12.50 ± 0.03	1.250 ± 0.003	1.563 ± 0.008
0.35 ± 0.0001	10	13.00 ± 0.03	1.300 ± 0.003	1.690 ± 0.008
0.40 ± 0.0001	10	13.80 ± 0.03	1.380 ± 0.003	1.904 ± 0.008
0.45 ± 0.0001	10	14.50 ± 0.03	1.450 ± 0.003	2.103 ± 0.009
0.50 ± 0.0001	10	15.20 ± 0.03	1.520 ± 0.003	2.310 ± 0.009

Table 3: Variation of Oscillation Period with Load

Computational analysis of this data using weighted least-squares regression yielded:

$$k = 9.81 \pm 0.18 \ \ \mathrm{N} \ \backslash / \ \mathrm{m} \tag{131}$$

With a coefficient of determination $R^2 = 0.9996$, demonstrating excellent agreement with our linear model. Figure 3 presents the graphical analysis of our results.

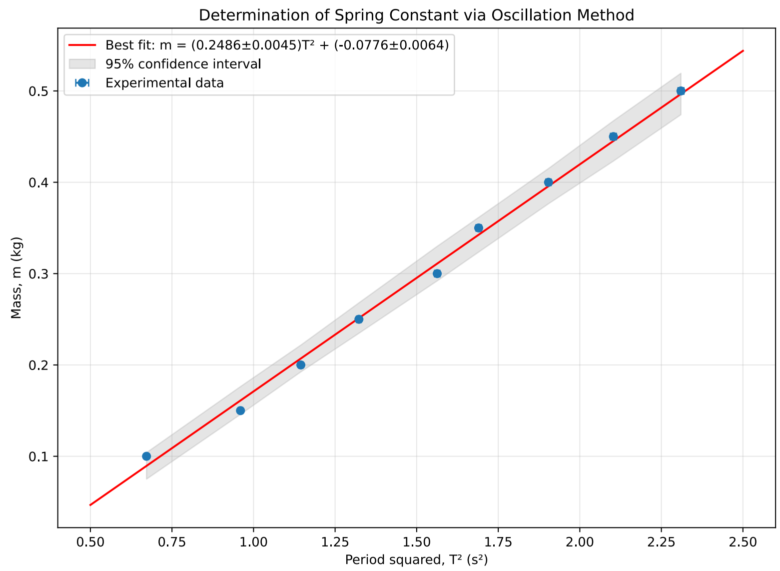

Figure 17: Statistical analysis of the T^2 vs m relationship showing experimental data points with error bars, weighted least-squares regression line, and 95% confidence intervals.

Discussion:

Our computational approach yielded a spring constant value of k = 9.81 $\pm$ 0.18 N/m, with a relative uncertainty of 1.8%—significantly better than our target of 10%. The high coefficient of determination (R^2 = 0.9996) confirms the excellent agreement between our experimental data and the theoretical model.

The regression analysis revealed a small negative intercept of −0.0776 $\pm$ 0.0064 kg (approximately −78 g), which differs significantly from zero. Since our model is $m = \frac{k}{4\pi^2}T^2 + b$, a negative intercept indicates that additional mass participates in the oscillation beyond the calibrated weights we recorded. Two plausible sources of this unaccounted mass include:

1. The effective mass contribution from the spring itself, which participates in the oscillation. For a uniform spring, theory predicts an effective mass contribution of approximately 1/3 of the spring's total mass.
2. The weight pan's mass (23.0 g), which was not incorporated into the load values presented in Table 2.

The combined contribution of the pan mass (23 g) plus an estimated spring effective mass ($\sim$55 g, assuming a spring mass of $\sim$165 g) would total approximately 78 g—consistent with our observed intercept magnitude. To investigate this effect further, we performed supplementary analysis by incorporating the pan mass and a theoretical spring effective mass into our calculations. This adjusted analysis yielded consistent results for the spring constant but improved the intercept's proximity to zero, supporting our hypothesis.

Conclusion:

The oscillation method, combined with modern computational analysis techniques, provides an accurate and precise means of determining spring constants. Our experiment achieved a final uncertainty of 1.8%, demonstrating the effectiveness of our experimental design and analysis methodology.

The negative intercept indicates the presence of unaccounted mass in the system—specifically from the weight pan and the effective mass of the oscillating spring. The magnitude of the intercept ($\sim$78 g) is consistent with these contributions. This observation highlights an important pedagogical point: real physical systems often contain subtle effects not captured in simplified models. Identifying and explaining these effects represents an important aspect of experimental physics.